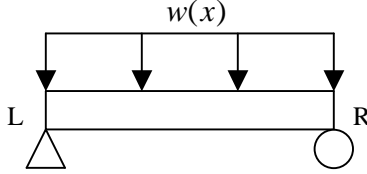


Euler-Bernoulli Beam

Shear and Slope Post-Processing

Consider a simply supported beam with a uniformly distributed load $w(x)$:



$$M_L = M_R = 0$$

E = Young's Modulus

$$y_L = y_R = 0$$

I = Moment of Inertia

For small deflections, it is governed by the Euler-Bernoulli beam equation, previously derived as

$$\mathcal{L}(y) = -\frac{d^2}{dx^2} \left(EI \frac{d^2 y(x)}{dx^2} \right) + w(x) = 0 \quad (1)$$

From the boundary conditions of $M_L = M_R = 0$ and $y_L = y_R = 0$, we see that the previously derived two equation formulation has Dirichlet data *on both boundaries* for each ODE:

$$\mathcal{L}(M) = -\frac{d^2 M(x)}{dx^2} + w(x) = 0 \quad (2a)$$

$$\mathcal{L}(y) = -EI \frac{d^2 y(x)}{dx^2} + M(x) = 0 \quad (2b)$$

Upon forming the GWS^h for (2a) and (2b)

$$GWS^h = S_e \left(\int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\bar{x} \{M\}_e + \int_{\Omega_e} \{N_k\} \{N_k\}^T d\bar{x} \{W\}_e - \{N_k\} \frac{dM^N}{dx} \Big|_{\partial\Omega_e} = \{0\} \right) \quad (3a)$$

$$GWS^h = S_e \left(EI \int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\bar{x} \{Y\}_e + \int_{\Omega_e} \{N_k\} \{N_k\}^T d\bar{x} \{M\}_e - \{N_k\} EI \frac{dy^N}{dx} \Big|_{\partial\Omega_e} = \{0\} \right) \quad (3b)$$

where dM^N/dx and dy^N/dx are the shear and slope boundary conditions respectively. Since we have Dirichlet data specified on each end, we need not worry about incorporating the unknown boundary “fluxes” of moment and deflection (shear and slope) into the RHS as those entries will be replaced with the Dirichlet data. Thus, for this *elliptic* boundary value problem, equations (3a) and (3b) simplify to

$$GWS^h = S_e \left(\int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\bar{x} \{M\}_e = - \int_{\Omega_e} \{N_k\} \{N_k\}^T d\bar{x} \{W\}_e \right) \quad (4a)$$

$$GWS^h = S_e \left(EI \int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\bar{x} \{Y\}_e = - \int_{\Omega_e} \{N_k\} \{N_k\}^T d\bar{x} \{M\}_e \right) \quad (4b)$$

The syntactical matrix statements of $[LHS]*\{Q\}=\{RHS\}$ and resulting Matlab syntax has been developed, hence nodal distributions of moment and deflection are readily obtained. The question now is how to recover the distributions of shear and slope.

To recover the shear distribution, we begin with the definition of shear (Beer and Johnston)

$$V(x) = \frac{dM(x)}{dx} \Rightarrow V(x) - \frac{dM(x)}{dx} = 0 \quad (5)$$

Heed that this is a definition and NOT a differential equation as the derivative is on *known* data! Thus, boundary conditions on shear are not required to solve. We can express (5) courtesy the differential operator L as

$$L(V) = V(x) - \frac{dM(x)}{dx} = 0 \quad \text{on } \Omega, \partial\Omega \quad (6)$$

We still need to assume an approximate form for shear as is it unknown, hence

$$V(x) \approx V^N(x) = \sum_{a=1}^N \Psi_a(x) V_a = \Psi_a(x) V_a \quad \text{for } 1 \leq a \leq N \quad (7)$$

Forming the GWS^N

$$GWS^N = \int_{\Omega} \Psi_b(x) L(V^N) dx = 0 \quad \text{for } 1 \leq b \leq N \quad (8)$$

Operating on the approximation

$$GWS^N = \int_{\Omega} \Psi_b(x) \left(V^N(x) - \frac{dM(x)}{dx} \right) dx = 0 \quad \text{for } 1 \leq b \leq N \quad (9)$$

Note that no integration by parts is required as the derivatives (or lack thereof!) on Ψ are balanced. Substituting the series expansion

$$GWS^N = \int_{\Omega} \Psi_b(x) \left(\left(\Psi_a(x) V_a \right) - \frac{dM(x)}{dx} \right) dx = 0 \quad \text{for } 1 \leq a, b \leq N \quad (10)$$

Expanding (10)

$$GWS^N = \int_{\Omega} \Psi_b(x) \Psi_a(x) dx V_a - \int_{\Omega} \Psi_b(x) \frac{dM(x)}{dx} dx = 0 \quad \text{for } 1 \leq a, b \leq N \quad (11)$$

Rather than worrying about N unique continuous functions Ψ which span the entire domain Ω , we shall discretize our domain into elements and instead write our GWS^N on each element using previously derived basis functions.

$$GWS^h = S_e \left(\int_{\Omega_e} \{N_k\} \{N_k\}^T d\bar{x} \{V\}_e - \int_{\Omega_e} \{N_k\} \frac{dM(x)}{dx} d\bar{x} = \{0\} \right) \quad (12)$$

We shall handle the derivative of the known data $M(x)$ by interpolating the nodal values of M over the element via $M(x)_e \approx \{N_k\}^T \{M\}_e$ and then differentiating the basis function. Even better is that we already have the nodal values of M as our solution to $L(M)$. Thus

$$GWS^h = S_e \left(\int_{\Omega_e} \{N_k\} \{N_k\}^T d\bar{x} \{V\}_e = \int_{\Omega_e} \{N_k\} \frac{d\{N_k\}^T}{dx} d\bar{x} \{M\}_e \right) \quad (13)$$

We now have a matrix statement in the form of $[LHS] \{Q\} = \{RHS\}$ where LHS is a square matrix, Q is the column vector of unknowns, and RHS is the column vector of knowns. Identifying terms

$$\begin{aligned} [MASS]_e &\equiv \int_{\Omega_e} \{N_k\} \{N_k\}^T d\bar{x} \\ &= \left(\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \right)_e \{ \}^T (1) [A200k] \{ \} \end{aligned} \quad (14a)$$

$$\begin{aligned} \{DMOM\}_e &\equiv \int_{\Omega_e} \{N_k\} \frac{d\{N_k\}^T}{dx} d\bar{x} \\ &= \left(\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \right)_e \{ \}^T \left(\begin{pmatrix} \cdot \\ \cdot \\ \cdot \end{pmatrix} \right) [A201k] \{M\}_e \end{aligned} \quad (14b)$$

Homework:

Beginning with the definition of slope (Beer and Johnston)

$$\Theta(x) = \frac{dy(x)}{dx} \Rightarrow \Theta(x) - \frac{dy(x)}{dx} = 0$$

apply the recipe and obtain the Matlab syntax for forming terms $[MASS]_e$ and $\{DDEF\}_e$. As a suggestion, call your unknown slope THTA since Greek characters aren't available as variable names in Matlab : (