Euler-Bernoulli Beam

From *Mechanics of Materials* by Beer and Johnston, section 8.3 derives the small deflection of a beam as a function of the moment distribution to be (equation 8.4 in the text)

$$M(x) = EI(x)\frac{d^2 y(x)}{dx^2} \tag{1}$$

Section 7.4 relates shear to applied distributed loads (equation 7.3 in the text) and moment to shear (equation 7.5 in the text) via

$$\frac{dV(x)}{dx} = w(x) \tag{2}$$

$$\frac{dM(x)}{dx} = V(x) \tag{3}$$

Note that Beer and Johnston employ a $-w(x)$ in (2) because they have assumed downward loads to be positive. We shall assume upwards loads to be positive, hence the $w(x)$ term in (2) is positive.

Differentiating (3) with respect to x and substituting into (2) yields

$$\frac{d^2 M(x)}{dx^2} = \frac{dV(x)}{dx} = w(x) \tag{4}$$

Differentiating (1) twice and substituting into (4)

$$\frac{d^2 M(x)}{dx^2} = \frac{d^2}{dx^2}\left(EI(x)\frac{d^2 y(x)}{dx^2}\right) = w(x) \tag{5}$$

Thus, the Euler-Bernoulli beam, after a multiplication by -1, equation takes the form of

$$\mathrm{L}(y) = -\frac{d^2}{dx^2}\left(EI(x)\frac{d^2 y(x)}{dx^2}\right) + w(x) = 0 \quad \Omega \in (0, L) \tag{6}$$

Heed that this is a fourth order differential equation, hence four boundary conditions must be applied. These boundary conditions can be Dirichlet (applied deflection), Neumann (applied slope), or higher order via applied shear or applied moment.

Applying the recipe to (6)

$$y(x) \approx y^N(x) = \Psi_a(x) Q_a \quad \text{for} \quad 1 \le a \le N \tag{7}$$

$$GWS^N = \int_\Omega \Psi_b \, \mathrm{L}(y^N) dx = \int_\Omega \Psi_b \left(-\frac{d^2}{dx^2}\left(EI(x)\frac{d^2 y^N}{dx^2}\right) + w(x)\right) dx \tag{8}$$

Noting that the trial function must support four derivatives, we can look ahead and recognize that, upon discretization, our basis function must be at least a fourth degree polynomial! Integrating by parts to reduce the required degree of the basis functions

$$GWS^N = \int_\Omega \frac{d\Psi_b}{dx}\frac{d}{dx}\left(EI(x)\frac{d^2 y^N}{dx^2}\right) dx + \int_\Omega \Psi_b \, w(x) dx - \Psi_b \, \frac{d}{dx}\left(EI(x)\frac{d^2 y^N}{dx^2}\right)\Bigg|_{\partial\Omega} \tag{9}$$

Excellent - the degree has been reduced to three and we have introduced the shear on the boundary. We could proceed and integrate by parts again, thereby symmeterizing the derivatives, reducing the required basis order to two, and introducing moment on the boundary, but we don't have any master matrices at our disposal to handle this result. Additionally, no natural mechanism for imposing deflection as a boundary condition would exist - a serious detriment to the formulation.

As a different approach, let's break (6) down into *two* differential equations, connected through the definition of moment. We can write a differential equation on the moment by substituting (1) into (6), hence

$$\mathrm{L}(M) = -\frac{d^2 M(x)}{dx^2} + w(x) = 0 \qquad (10a)$$

Thus we can solve (10a) for moment distribution knowing the distributed load $w(x)$. Having solved for $M(x)$, we can solve for the deflection by substituting it into (1), hence

$$\mathrm{L}(y) = -EI(x)\frac{d^2 y(x)}{dx^2} + M(x) = 0 \qquad (10b)$$

We now seek to apply the "recipe" to equations (10a) and (10b) and end up with the necessary Matlab syntax for numeric implementation. Beginning with the differential equation written on *M*:

$$\mathrm{L}(M) = -\frac{d^2 M(x)}{dx^2} + w(x) = 0 \qquad \Omega \in (0, L) \qquad (10a)$$

Generating the continuous approximation to the unknown exact solution $M(x)$

$$M(x) \approx M^N(x) = \sum_{a=1}^{N} \Psi_a(x) M_a = \Psi_a(x) M_a \quad for \quad 1 \le a \le N \qquad (11)$$

Writing the Galerkin Weak Statement

$$GWS^N = \int_\Omega \Psi_b(x)\, \mathrm{L}(M^N)\, dx = 0 \qquad 1 \le b \le N \qquad (12)$$

Substituting (10a) into (12)

$$GWS^N = \int_\Omega \Psi_b(x)\left(-\frac{d^2 M^N(x)}{dx^2} + w(x)\right) dx = 0 \qquad 1 \le b \le N \qquad (13)$$

Integrating by parts

$$GWS^N = \int_\Omega \frac{d\Psi_b(x)}{dx}\frac{dM^N(x)}{dx}\, dx + \int_\Omega \Psi_b(x)\, w(x)\, dx - \Psi_b \left.\frac{dM^N}{dx}\right|_{\partial\Omega} = 0 \qquad 1 \le b \le N \qquad (14)$$

Substituting (11) into (14)

$$GWS^N = \int_\Omega \frac{d\Psi_b(x)}{dx}\frac{d\Psi_a(x)}{dx}\, dx\, M_a + \int_\Omega \Psi_b(x)\, w(x)\, dx - \Psi_b \left.\frac{dM^N}{dx}\right|_{\partial\Omega} = 0 \qquad 1 \le a, b \le N \qquad (15)$$

Discretizing the domain

$$\Psi_a(x)M_a = \mathrm{S}_e\left(\{N_k(\zeta_i)\}^T\{M\}_e\right) \tag{16}$$

and forming GWS^h

$$GWS^h = \mathrm{S}_e\left(\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}\,d\bar{x}\{M\}_e + \int_{\Omega_e}\{N_k\}\{N_k\}^T d\bar{x}\{W\}_e - \{N_k\}\frac{dM^N}{dx}\bigg|_{\partial\Omega} = \{0\}\right) \tag{17}$$

$$GWS^h = \mathrm{S}_e\left(\int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}\,d\bar{x}\{M\}_e = -\int_{\Omega_e}\{N_k\}\{N_k\}^T d\bar{x}\{W\}_e + \{N_k\}\frac{dM^N}{dx}\bigg|_{\partial\Omega}\right) \tag{18}$$

We now have a matrix statement of the form LHS*Q=RHS where LHS is a square matrix, Q is the column vector of unknowns, and RHS is the column vector of knowns. Identifying the terms

$$\begin{aligned}[\mathrm{DIFF}]_e &\equiv \int_{\Omega_e}\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}\,d\bar{x} \\ &= (\ )(\ )_e\ \{\ \}_e^T\ (-1)[\mathrm{A211k}]\{\ \}_e\end{aligned} \tag{19a}$$

$$\begin{aligned}\{\mathrm{LOAD}\}_e &\equiv \int_{\Omega_e}\{N_k\}\{N_k\}^T d\bar{x}\{W\}_e \\ &= (\ )(\ )_e\ \{\ \}_e^T\ (1)[\mathrm{A200k}]\{W\}_e\end{aligned} \tag{19b}$$

$$\{\mathrm{BSHR}\}_e \equiv \{N_k\}\frac{dM^N}{dx}\bigg|_{\partial\Omega} \tag{19c}$$

The boundary conditions for solving are shear (Neumann) on the boundary (courtesy integration by parts) and moment (Dirichlet) on the boundary.

Having solved for the distribution of moment, we can solve (10b) for the deflection. We will initially assume the mass moment of inertia to be constant. Beginning with the differential equation written on *y*:

$$\mathrm{L}(y) = -EI\frac{d^2y(x)}{dx^2} + M(x) = 0 \tag{10b}$$

Generating the continuous approximation to the unknown exact solution $y(x)$

(11)

Writing the Galerkin Weak Statement

(12)

Substituting (10b) into (12)

(13)

Integrating by parts

(14)

Substituting (11) into (14)

(15)

Discretizing the domain

(16)

and forming GWS^h

(17)

(18)

We now have a matrix statement of the form LHS*Q=RHS where LHS is a square matrix, Q is the column vector of unknowns, and RHS is the column vector of knowns. Identifying the terms

(19a)

(19b)

(19c)

The boundary conditions for solving are slope (Neumann) on the boundary (courtesy integration by parts) and deflection (Dirichlet) on the boundary.