# Steady State Conduction with Boundary Convection

Let us now modify our one-dimensional, steady-state heat conduction model by replacing the applied heat flux boundary condition with a convective heat transfer, hence

$$\mathrm{L}(T) = -\frac{d}{dx}\left(k(x)\frac{dT}{dx}\right) - s(x) = 0 \qquad \Omega \in (0, L) \tag{1a}$$

$$\mathrm{l}(T) = k(x)\frac{dT}{dn} + h(T - T_r) = 0 \qquad \partial\Omega_1 \in 0 \tag{1b}$$

$$T = T_b \qquad \partial\Omega_2 \in L \tag{1c}$$

**Step 1:** Assume an approximate solution exists

$$T(x) \approx T^N(x) = \sum_{\alpha=1}^{N} \Psi_\alpha(x)\, Q_\alpha = \Psi_\alpha(x)\, Q_\alpha \quad \text{for} \quad 1 \le \alpha \le N \tag{2}$$

**Step 2:** What are suitable trial functions and how do we solve for expansion coefficients?

**Step 3:** Solve for expansion coefficients by

a. Defining the error in the approximation

$$e^N(x) = T(x) - T^N(x) \tag{3a}$$

b. Measuring the error

$$\mathrm{L}(e^N) = \mathrm{L}(T) - \mathrm{L}(T^N) = -\mathrm{L}(T^N) \tag{3b}$$

c. Minimizing the error

$$WS^N = \int_\Omega \Phi_\beta(x)\, \mathrm{L}(T^N)\, dx \equiv 0 \qquad \text{for} \quad 1 \le \beta \le N \tag{3c}$$

d. Optimizing the minimization

$$GWS^N = \int_\Omega \Psi_\beta(x)\, \mathrm{L}(T^N)\, dx \equiv 0 \qquad \text{for} \quad 1 \le \beta \le N \tag{3d}$$

e. Applying the operator

$$GWS^N = \int_\Omega \Psi_\beta(x)\left(-\frac{d}{dx}\left(k(x)\frac{dT^N}{dx}\right) - s(x)\right) dx = 0 \tag{3e}$$

f. Integrate by parts

$$GWS^N = \int_\Omega \frac{d\Psi_\beta(x)}{dx} k(x) \frac{dT^N}{dx}\, dx - \int_\Omega \Psi_\beta(x) s(x)\, dx - \Psi_\beta(x) k(x) \frac{dT^N}{dx}\bigg|_{\partial\Omega} = 0 \qquad \text{for} \quad 1 \le \beta \le N$$

We must now pay close attention to our boundary conditions before moving forward.

Expanding the boundary condition term

$$k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega} = \left(+k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega_2}\right) - \left(-k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega_1}\right)$$
$$= k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega_2} - h\left(T^N - T_r\right)\big|_{\partial\Omega_1}$$

Substituting back

$$GWS^N = \int_\Omega \frac{d\Psi_b(x)}{dx}k(x)\frac{dT^N}{dx}dx - \int_\Omega \Psi_b(x)s(x)dx + \Psi_b(x)h\left(T^N - T_r\right)\big|_{\partial\Omega_1} - \Psi_b(x)k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega_2} = 0 \qquad (3f)$$

g. Substitute the series expansion

$$GWS^N = \int_\Omega \frac{d\Psi_b(x)}{dx}k(x)\frac{d\Psi_a(x)}{dx}dxQ_a - \int_\Omega \Psi_b(x)s(x)dx + \Psi_b(x)h\left(\Psi_a(x)Q_a - T_r\right)\big|_{\partial\Omega_1} - \Psi_b(x)k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega_2} = 0$$

**Step 4:** Select Lagrange polynomials as our trial functions

**Step5:** Discretize the problem domain

$$\Psi_a(x)Q_a = S_e\left(\{N_k(z_i)\}^T\{Q\}_e\right) \qquad (5a)$$
$$\Psi_b = S_e\left(\{N_k(z_i)\}\right) \qquad (5b)$$

**Step 6:** Form discrete $GWS^h$

$$GWS^h = S_e\left(\int_{\Omega_e} \frac{d\{N_k\}}{dx}k(x)\frac{d\{N_k\}^T}{dx}d\bar{x}\{Q\}_e - \int_{\Omega_e}\{N_k\}s(x)d\bar{x} + \{N_k\}h\left(\{N_k\}^T\{Q\}_e - T_r\right)\big|_{\partial\Omega_1} - \{N_k\}k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega_2} = \{0\}\right)$$

a. Get all the unknowns on the left

$$GWS^h = S_e\left(\int_{\Omega_e} k(x)\frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}d\bar{x}\{Q\}_e + h\{N_k\}\{N_k\}^T\{Q\}_e\big|_{\partial\Omega_1} = \int_{\Omega_e}\{N_k\}s(x)d\bar{x} + hT_r\{N_k\}\big|_{\partial\Omega_1} + \{N_k\}k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega_2}\right)$$

b. Give each term a name and decide what to do with distributed and boundary terms

Integrated terms:

$$
\begin{aligned}
[\text{DIFF}]_e &\equiv \int_{\Omega_e} k(x) \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\overline{x} \\
&\approx \{K\}_e^T \int_{\Omega_e} \{N_k\} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\overline{x} \\
&= (\ )(\ )_e \ \{K\}_e^T \ (-1)[\text{A3011k}]\{\ \}_e
\end{aligned}
\tag{6a}
$$

$$
\begin{aligned}
\{\text{SRC}\}_e &\equiv \int_{\Omega_e} \{N_k\} s(x) d\overline{x} \\
&\approx \int_{\Omega_e} \{N_k\}\{N_k\}^T d\overline{x} \{S\}_e \\
&= (\ )(\ )_e \ \{\ \}_e^T \ (1)[\text{A200k}]\{S\}_e
\end{aligned}
\tag{6b}
$$

Boundary terms:

$$
[\text{HBC}]_e \equiv h\{N_k\}\{N_k\}^T \{Q\}_e \Big|_{\partial\Omega_1} \tag{6c}
$$

$$
\{\text{HTR}\}_e \equiv hT_r \{N_k\}\Big|_{\partial\Omega_1} \tag{6d}
$$

$$
\{\text{BFLX}\}_e \equiv \{N_k\} k(x) \frac{dT^N}{dx}\Bigg|_{\partial\Omega_2} \tag{6e}
$$

What new stuff do we have?

**Interpolated thermal conductivity**

Examining the three bases within the integral for $[\text{DIFF}]_e$ , we see an *incorrect* matrix multiplication!

$$
\{K\}_e^T \int_{\Omega_e} \{N_k\} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\overline{x}
$$

The undifferentiated basis, however, multiplies the (constant) nodal values of conductivity and yields a scalar on the element $\{K\}_e^T \{N_k\}$. Assuming a linear basis, $[\text{DIFF}]_e$ becomes

$$
\{K\}_e^T \int_{\Omega_e} \{N_1\} \frac{1}{l_e^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\overline{x}
$$

Distributing the undifferentiated basis yields the *hypermatrix*

$$
\{K\}_e^T \int_{\Omega_e} \frac{1}{l_e^2} \begin{bmatrix} \{N_1\} & -\{N_1\} \\ -\{N_1\} & \{N_1\} \end{bmatrix} d\overline{x}
$$

where it is understood that $\{K\}_e^T$ multiplies every value of $\{N_1\}$

**Boundary terms**

Drawing a simple, three-element domain, we note that boundary conditions are applied exactly on the boundaries. They are not interpolated over the interior. Our $GWS^h$ boundary terms $[\mathrm{HBC}]_e$, $\{\mathrm{HTR}\}_e$, and $\{\mathrm{BFLX}\}_e$ must therefore employ a physically consistent, reduced basis function. Sticking with the linear basis for demonstration, the distribution of basis functions becomes

Examining the leftmost element, the leftmost boundary conditions are applied via $z_1 = 1$ and $z_2 = 0$ at the leftmost node. The leftmost boundary conditions are applied nowhere else, hence $z_1 = z_2 = 0$ on all other nodes.

Examining the rightmost element, the rightmost boundary conditions are applied via $z_1 = 0$ and $z_2 = 1$ at the rightmost node. The rightmost boundary conditions are applied nowhere else, hence $z_1 = z_2 = 0$ on all other nodes.

Hence, on the boundaries, our basis functions reduce to a binary switch - "on" at the appropriate node and "off" at all other nodes. This is conveniently expressed with the Kronecker delta function

$$d_{ij} = 1 \quad \text{for} \quad i = j$$
$$d_{ij} = 0 \quad \text{for} \quad i \neq j$$

The boundary basis functions thus become

**linear**

$$\{N_1\}\big|_{\partial\Omega} \equiv \begin{Bmatrix} d_{e1} \\ d_{eM} \end{Bmatrix}$$

**quadratic**

$$\{N_2\}\big|_{\partial\Omega} \equiv \begin{Bmatrix} d_{e1} \\ 0 \\ d_{eM} \end{Bmatrix}$$

**cubic**

$$\{N_3\}\big|_{\partial\Omega} \equiv \begin{Bmatrix} d_{e1} \\ 0 \\ 0 \\ d_{eM} \end{Bmatrix}$$

Thus, the square matrix $[\text{HBC}]_e$ will "assemble" to an nnodes *x* nnodes matrix of zeros with the convective heat transfer coefficient $h$ in either the (1,1) position (left BC) or the (nnodes, nnodes) position (right BC). The column vector $\{\text{HTR}\}_e$ will assemble to an nnodes *x* 1 column of zeros with reference heat transfer $hT_r$ in either the (1,1) position or the (nnodes, 1) position.

Why can we ignore the column vector $\{\text{BFLX}\}_e$?