Formal Accuracy and Convergence

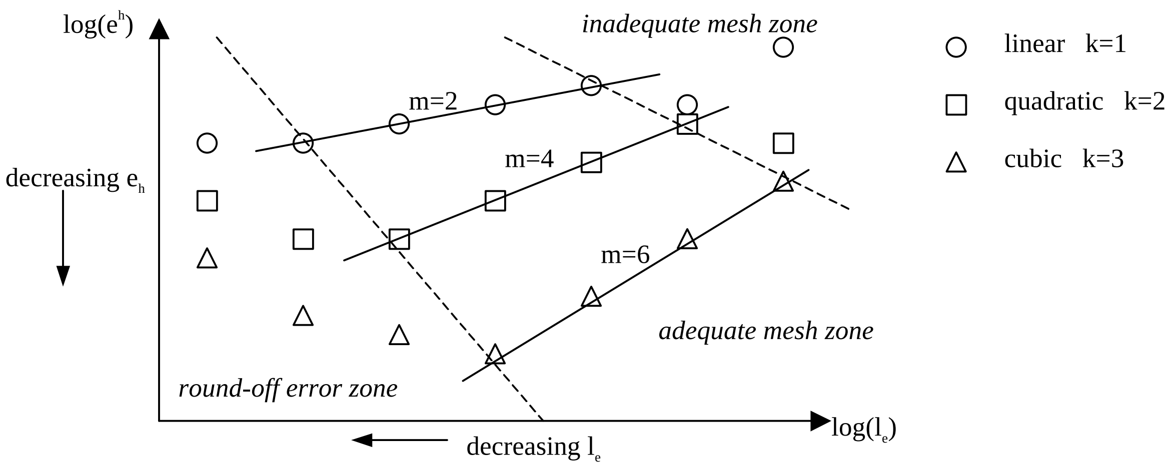
We shall accept as truth that the approximation error, under uniform mesh refinement, will behave according to

$$e^h = C_k l_e^{2k} \tag{1}$$

Taking the base 10 log of both sides we arrive at the equation for a straight line

$$\log\left(e^h\right) = 2k\log(l_e) + \log(C_k) \tag{2}$$

Thus, the log of the error in our approximation is a linear function of the log of the element length. Be sure to note that the slope of the line is two times the basis degree. Heed that the y-intercept $\log(C_k)$, however, is both basis and problem dependent and hence a unique unknown for every problem. Assuming we have an exact solution to compare with our finite element approximate solutions, we would get a plot which looks like



The existence of (1), courtesy Oden and Reddy and verified by Baker and Soliman, is of singular importance in that it allows to quantitatively estimate the error in an approximation. Specifically, for one mesh and its double refinement Ω^h and $\Omega^{h/2}$, the corresponding approximate solutions Q^h and $Q^{h/2}$ satisfy the error definition

$$e^h = Q_{exact} - Q^h$$

hence

$$Q^h + e^h = Q_{exact} = Q^{h/2} + e^{h/2} \tag{3}$$

Letting $l_e = h$ and $l_e = h/2$. (1) becomes

$$e^h = \left(2^{2k}\right)e^{h/2} \tag{4}$$

provided that Q^h and $Q^{h/2}$ lie on the linear convergence curve. This implies that the initial mesh must be fine enough to have a “good” wrong answer. The key feature of (4) is that it is independent of the (unknown problem dependent) constant C_k present in (1). Substituting (4) into (3) then yields

$$Q^{h/2} - Q^h = \left(2^{2k} - 1\right)e^{h/2} \tag{5}$$

Letting $\Delta Q^{h/2} \equiv Q^{h/2} - Q^h$ denote the computed difference in the approximate solutions on the two meshes, the approximate error in the finer grid solution is

$$e^{h/2} = \frac{\Delta Q^{h/2}}{2^{2k} - 1} \tag{6}$$

Equation (6) is a quantitative error estimate independent of C_k, and is valid anywhere on Ω provided the approximate solutions Q^h and $Q^{h/2}$ lie on the convergence curve. You may easily confirm whether the "adequate mesh" requirement is satisfied by evaluating (6) with two pairs of refined mesh solutions, i.e. for Ω^h, $\Omega^{h/2}$, and $\Omega^{h/4}$. Then, if the resultant two data points $e^{h/2}$ and $e^{h/4}$ lie on a straight line of slope $2k$ on a log-log plot,

$$slope^{h/4} = \frac{\log\left(\frac{e^{h/2}}{e^{h/4}}\right)}{\log(2)} \approx 2k \tag{7}$$

all solutions Q^h, $Q^{h/2}$, and $Q^{h/4}$ satisfy the adequate mesh requirement. As an added bonus, should the approximate solution satisfy the adequate mesh requirement, the *exact solution* can be estimated via (3).

$$Q_{exact} \approx Q^{h/2} + e^{h/2} \tag{8}$$

A slight theoretical flaw exists in this development to which the purist should take exception. Specifically, (1) may not always hold at every location x within Ω, especially where material properties change abruptly, e.g. a thermal conductor-insulator interface or on a boundary. However, (1) will be consistently reliable is one selects the mathematician's expression for "error" which elegantly includes all such issues. One such error measure is the "energy" in the solution, denoted the "energy seminorm" and is defined as

$$\left\| Q^h \right\|_E \equiv \frac{1}{2} S_e \left(\{Q\}_e^T \left[\mathrm{DIFF} + \mathrm{HBC}\right]_e \{Q\}_e \right) \tag{9}$$

where $[\mathrm{DIFF}]_e$ is the element level conduction matrix and $[\mathrm{HBC}]_e$ is the element level convective boundary condition matrix. For approximation error measured in the energy seminorm, i.e.

$$\left\| e^h \right\|_E \equiv \left\| T_{exact} \right\|_E - \left\| T^h \right\|_E \tag{10}$$

the revised form of the estimate (6) becomes

$$\left\| e^{h/2} \right\|_E = \frac{\Delta \left\| Q^{h/2} \right\|_E}{2^{2k} - 1} \tag{11}$$

Note that the formation of the energy seminorm is quite simple courtesy the syntactical GWS^h and Matlab assembly routines.

Homework:

1. Starting with (1), fill in the algebraic detail to obtain (6) and (7)