Higher Order Bases

The accuracy of a finite element solution can be improved by either increasing the mesh (*h*-refinement) or by increasing the degree of the basis function (*p*-refinement). While this course will focus primarily on *h*-refinement, considerable insight can be gained by briefly investigating *p*-refinement.

For our study problem of one-dimensional, steady-state heat conduction with variable conductivity and source, the recipe lead to

$$GWS^h = S_e GWS_e^h = S_e\left(\int_{\Omega_e} \frac{d\{N_k\}}{dx} k(x) \frac{d\{N_k\}^T}{dx}\, d\bar{x} \{Q\}_e - \int_{\Omega_e} \{N_k\} s(x) d\bar{x} - k(x)\frac{dT^N}{dx}\begin{Bmatrix} d_{e1} \\ d_{eM} \end{Bmatrix} = \{0\}_e \right) \tag{1a}$$

which was expressed compactly as

$$GWS^h = S_e GWS_e^h = S_e\left([\mathrm{DIFF}]_e \{Q\}_e = \{\mathrm{SRC}\}_e + \{\mathrm{BFLX}\}_e \right) \tag{1b}$$

We must now evaluate the matrix integrals leading to $[\mathrm{DIFF}]_e$ and $\{\mathrm{SRC}\}_e$. Let us handle the conductivity as an element averaged term and the source as an element interpolated term. Hence

$$\begin{aligned} [\mathrm{DIFF}]_e \{Q\}_e &= \int_{\Omega_e} \frac{d\{N_k\}}{dx} k(x) \frac{d\{N_k\}^T}{dx}\, d\bar{x}\{Q\}_e \\ &\approx \bar{k}_e \int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx}\, d\bar{x}\{Q\}_e \\ &= \ (\ )\left(\bar{k}\right)_e \{\ \}_e^T \ (-1)\,[\mathrm{A211k}]\,\{Q\}_e \end{aligned} \tag{2a}$$

$$\begin{aligned} \{\mathrm{SRC}\}_e &= \int_{\Omega_e} \{N_k\} s(x) d\bar{x} \\ &\approx \int_{\Omega_e} \{N_k\}\{N_k\}^T d\bar{x}\{S\}_e \\ &= (\ )(\ )_e \{\ \}_e^T \ (1)\,[\mathrm{A200k}]\,\{S\}_e \end{aligned} \tag{2b}$$

Our earlier discussion on interpolation theory revealed the linear, quadratic, and cubic basis functions to take the form of

Linear:

$$\{N_1(z_i)\} = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} \tag{3a}$$

Quadratic:

$$\{N_2(z_i)\} \equiv \begin{Bmatrix} z_1(2z_1 - 1) \\ 4z_1 z_2 \\ z_2(2z_2 - 1) \end{Bmatrix} \tag{3b}$$

Cubic:

$$\{N_3(z_i)\} \equiv \frac{9}{2}\begin{Bmatrix} z_1\left(z_2^2 - z_2 + \frac{2}{9}\right) \\ z_1 z_2(2-3z_2) \\ z_1 z_2(3z_2 - 1) \\ z_2\left(z_2^2 - z_2 + \frac{2}{9}\right) \end{Bmatrix} \tag{3c}$$

where

$$z_1(\bar{x}) \equiv 1 - \frac{\bar{x}}{l_e} \qquad z_2(\bar{x}) \equiv \frac{\bar{x}}{l_e} \qquad \bar{x} = x - X_L \tag{3d}$$

Let us begin with the linear basis function. Carefully applying the chain rule

$$\frac{d\{N_1(z_i)\}}{dx} = \frac{d\{N_1(z_i)\}}{dz_i}\frac{dz_i}{d\bar{x}}\frac{d\bar{x}}{dx} \qquad \text{for} \quad 1 \le i \le 2 \tag{4}$$

to the first entry

$$\begin{aligned} \frac{d\{N_1(1,1)\}}{dx} &= \frac{dz_1}{dz_1}\frac{dz_1}{d\bar{x}}\frac{d\bar{x}}{dx} + \frac{dz_1}{dz_2}\frac{dz_2}{d\bar{x}}\frac{d\bar{x}}{dx} \\ &= 1\left(-\frac{1}{l_e}\right)1 + 0 \\ &= -\frac{1}{l_e} \end{aligned} \tag{5a}$$

and to the second entry

$$\begin{aligned} \frac{d\{N_1(2,1)\}}{dx} &= \frac{dz_2}{dz_1}\frac{dz_1}{d\bar{x}}\frac{d\bar{x}}{dx} + \frac{dz_2}{dz_2}\frac{dz_2}{d\bar{x}}\frac{d\bar{x}}{dx} \\ &= 0 + 1\left(\frac{1}{l_e}\right)1 \\ &= \frac{1}{l_e} \end{aligned} \tag{5b}$$

Thus

$$\boxed{\frac{d\{N_1(z_i)\}}{dx} = \frac{1}{l_e}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}} \tag{6}$$

Differentiating the quadratic basis function

$$\frac{d\{N_2(z_i)\}}{dx} = \frac{d\{N_2(z_i)\}}{dz_i}\frac{dz_i}{d\bar{x}}\frac{d\bar{x}}{dx} \qquad \text{for} \quad 1 \le i \le 2 \tag{7}$$

First entry

$$\begin{aligned}\frac{d\{N_2(1,1)\}}{dx} &= \frac{d}{dz_1}\left(z_1(2z_1-1)\right)\frac{dz_1}{d\bar{x}}\frac{d\bar{x}}{dx} + \frac{d}{dz_2}\left(z_1(2z_1-1)\right)\frac{dz_2}{d\bar{x}}\frac{d\bar{x}}{dx} \\ &= \left((2z_1-1)+2z_1\right)\left(-\frac{1}{l_e}\right)1+0 \\ &= \frac{1}{l_e}(z_2-3z_1)\end{aligned} \tag{8a}$$

Second entry

$$\begin{aligned}\frac{d\{N_2(2,1)\}}{dx} &= \frac{d}{dz_1}(4z_1z_2)\frac{dz_1}{d\bar{x}}\frac{d\bar{x}}{dx} + \frac{d}{dz_2}(4z_1z_2)\frac{dz_2}{d\bar{x}}\frac{d\bar{x}}{dx} \\ &= (4z_2)\left(-\frac{1}{l_e}\right)1+(4z_1)\left(\frac{1}{l_e}\right)1 \\ &= \frac{1}{l_e}4(z_1-z_2)\end{aligned} \tag{8b}$$

Third entry

$$\begin{aligned}\frac{d\{N_2(3,1)\}}{dx} &= \frac{d}{dz_1}\left(z_2(2z_2-1)\right)\frac{dz_1}{d\bar{x}}\frac{d\bar{x}}{dx} + \frac{d}{dz_2}\left(z_2(2z_2-1)\right)\frac{dz_2}{d\bar{x}}\frac{d\bar{x}}{dx} \\ &= 0+\left((2z_2-1)+2z_2\right)\left(\frac{1}{l_e}\right)1+0 \\ &= \frac{1}{l_e}(3z_2-z_1)\end{aligned} \tag{9}$$

Thus

$$\boxed{\frac{d\{N_2(z_i)\}}{dx} = \frac{1}{l_e}\begin{Bmatrix} z_2-3z_1 \\ 4(z_1-z_2) \\ 3z_2-z_1 \end{Bmatrix}} \tag{10}$$

It is left as a homework exercise that

$$\boxed{\frac{d\{N_3(z_i)\}}{dx} = \frac{9}{2l_e}\begin{Bmatrix} -3z_2^2 + 4z_2 - 11/9 \\ 9z_2^2 - 10z_2 + 2 \\ -9z_2^2 + 8z_2 - 1 \\ 3z_2^2 - 2z_2 + 2/9 \end{Bmatrix}} \tag{11}$$

Having evaluated the basis derivatives, let us proceed to the element integrals. Starting with the $[\mathrm{A211k}]$ master matrix within $[\mathrm{DIFF}]_e$

$$\int_{\Omega_e} \frac{d\{N_k\}}{dx}\frac{d\{N_k\}^T}{dx}\, d\overline{x}$$

we have, for the linear basis

$$\int_{\Omega_e} \frac{d\{N_1\}}{dx}\frac{d\{N_1\}^T}{dx}\, d\overline{x} = \int_{\Omega_e} \frac{1}{l_e}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\frac{1}{l_e}\{-1 \quad 1\}d\overline{x} = \int_{\Omega_e} \frac{1}{l_e^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\overline{x} \tag{12}$$

It is of considerable convenience that the vector product is constant, yielding immediate evaluation

$$\boxed{\int_{\Omega_e} \frac{d\{N_1\}}{dx}\frac{d\{N_1\}^T}{dx}\, d\overline{x} = \frac{1}{l_e}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}} \tag{13}$$

The quadratic basis is a bit more difficult

$$\int_{\Omega_e} \frac{d\{N_2\}}{dx}\frac{d\{N_2\}^T}{dx}\, d\overline{x} = \int_{\Omega_e} \frac{1}{l_e^2}\begin{bmatrix} (z_2 - 3z_1)^2 & (z_2 - 3z_1)4(z_1 - z_2) & (z_2 - 3z_1)(3z_2 - z_1) \\ 4(z_1 - z_2)(z_2 - 3z_1) & 16(z_1 - z_2)^2 & 4(z_1 - z_2)(3z_2 - z_1) \\ (3z_2 - z_1)(z_2 - 3z_1) & (3z_2 - z_1)4(z_1 - z_2) & (3z_2 - z_1)^2 \end{bmatrix} d\overline{x} \tag{14}$$

The required integrals are readily evaluated by substituting the definitions of $z_1(\overline{x})$ and $z_2(\overline{x})$ into (13) and letting a symbolic package such as Maple do the gruntwork. However, Maple has not always existed and, as an historical note, the following analytical solution is presented. Integrals over Ω_e of *all* polynomials in the z_i can be evaluated as

$$\int_{\Omega_e} z_1^p z_2^q\, d\overline{x} = l_e \frac{p!\,q!}{(1+p+q)!} \tag{15}$$

Irrespective of method, (14) integrates out to

$$\boxed{\int_{\Omega_e} \frac{d\{N_2\}}{dx}\frac{d\{N_2\}^T}{dx}\, d\overline{x} = \frac{1}{3l_e}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}} \tag{16}$$

Completing the analysis with the cubic basis, it is left as a homework exercise to verify that

$$\int_{\Omega_e} \frac{d\{N_3\}}{dx}\frac{d\{N_3\}^T}{dx}\,d\bar{x} = \frac{1}{40 l_e}\begin{bmatrix} 148 & -189 & 54 & -13 \\ & 432 & -297 & 54 \\ & & 432 & -189 \\ (\text{sym}) & & & 148 \end{bmatrix} \tag{17}$$

In conclusion, we have evaluated the matrix form of the $[\text{A211k}]$ master matrix for linear, quadratic, and cubic basis functions. Heed that the metric for each basis remains -1.

Turning our attention to the master matrix $[\text{A200k}]$ within $\{\text{SRC}\}_e$

$$\int_{\Omega_e} \{N_k\}\{N_k\}^T\,d\bar{x}$$

we have, for the linear basis

$$\int_{\Omega_e} \{N_1\}\{N_1\}^T\,d\bar{x} = \int_{\Omega_e} \begin{Bmatrix} \zeta_1 \\ \zeta_2 \end{Bmatrix}\{\zeta_1 \quad \zeta_2\}\,d\bar{x} = \int_{\Omega_e} \begin{bmatrix} \zeta_1^2 & \zeta_1\zeta_2 \\ \zeta_2\zeta_1 & \zeta_2^2 \end{bmatrix} d\bar{x} \tag{18}$$

which is readily evaluated via Maple or (15) to give

$$\int_{\Omega_e} \{N_1\}\{N_1\}^T\,d\bar{x} = \frac{l_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{19}$$

Likewise, the quadratic and cubic bases yield

$$\int_{\Omega_e} \{N_2\}\{N_2\}^T\,d\bar{x} = \frac{l_e}{30}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix} \tag{20}$$

$$\int_{\Omega_e} \{N_3\}\{N_3\}^T\,d\bar{x} = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix} \tag{21}$$

Heed that the metric for each basis remains 1.

Homework:

1. Verify equation (11), the derivative of $\{N_3\}$.
2. Verify equation (17), the element master matrix $[\text{A211C}]$.
3. Figure out equation (21), the element master matrix $[\text{A200C}]$.