The Recipe and Syntactical GWS
1-D SS Heat Conduction with Variable Conductivity and Source

Let us now go through the key recipe steps for a one-dimensional, steady-state heat conduction problem with a variable thermal conductivity and source term, an applied heat flux boundary condition, and a prescribed temperature boundary condition. The governing equation system and associated boundary conditions are

$$\mathcal{L}(T) = -\frac{d}{dx}\left(k(x)\frac{dT}{dx}\right) - s(x) = 0 \qquad \Omega \in (0, L) \tag{1a}$$

$$\ell(T) = k(x)\frac{dT}{dn} - q_{in} = 0 \qquad \partial\Omega \in 0 \tag{1b}$$

$$T = T_b \qquad \partial\Omega \in L \tag{1c}$$

Walking through the recipe:

Step 1
This ODE may not have an exact, closed form solution. We will therefore be so bold as to assume that an approximate solution exists, the form of which is given by the series expansion

$$T(x) \approx T^N(x) = \sum_{\alpha=1}^{N} \Psi_\alpha(x) Q_\alpha = \Psi_\alpha(x) Q_\alpha \quad \text{for} \quad 1 \le \alpha \le N \tag{2}$$

where the *trial functions* $\Psi_\alpha(x)$ are continuous functions which span the domain Ω and the *expansion coefficients* Q_α are undetermined constants.

Step 2
What are suitable trial functions?
How do we solve for the unknown expansion coefficients?

Step 3
We will solve for the expansion coefficients by defining the error, measuring it in some manner, and then minimizing the error measure. The error in the approximation, denoted $e^N(x)$ is defined as the difference between the exact solution and the approximation, hence

$$e^N(x) = T(x) - T^N(x) \tag{3}$$

While many manners of measurement exist, we shall choose to measure the error by having the governing differential equation $\mathcal{L}(\ )$ operate on the error function yielding

$$\mathcal{L}(e^N) = \mathcal{L}(T) - \mathcal{L}(T^N) = -\mathcal{L}(T^N) \tag{4}$$

since the differential operator on T is identically zero courtesy (1a). Thus the measure of error is directly proportional to the differential operator on the approximate solution. To drive this measure to exactly zero would be equivalent to finding the exact solution (which does not exist), hence it must be minimized by some other mechanism. Capitalizing upon the mathematician's weak statement, the error measure will be made to vanish in an overall integrated sense.

$$WS^N = \int_\Omega \Phi_\beta(x)\,\mathcal{L}\left(T^N\right)dx \equiv 0 \qquad \text{for} \quad 1 \le \beta \le N \tag{5}$$

The optimal choice for the *test function* $\Phi_\beta(x)$ is to apply the Galerkin criterion which states that the trial function should be used as the test function. This selection is optimal in that it generates the approximate solution closest to the exact solution. Thus

$$GWS^N = \int_\Omega \Psi_\beta(x)\,\mathcal{L}\left(T^N\right)dx \equiv 0 \qquad \text{for} \quad 1 \le \beta \le N \tag{6}$$

Applying the operator (1a),

$$GWS^N = \int_\Omega \Psi_\beta(x)\left(-\frac{d}{dx}\left(k(x)\frac{dT^N}{dx}\right) - s(x)\right)dx = 0 \tag{7}$$

integrating by parts to decrease the required differentiability order of the test function

$$GWS^N = \int_\Omega \frac{d\Psi_\beta(x)}{dx}k(x)\frac{dT^N}{dx}dx - \int_\Omega \Psi_\beta(x)s(x)dx - \Psi_\beta(x)k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega} = 0 \qquad \text{for} \quad 1 \le \beta \le N \tag{8}$$

and substituting the series expansion (3),

$$GWS^N = \int_\Omega \frac{d\Psi_\beta(x)}{dx}k(x)\frac{d\Psi_\alpha(x)}{dx}dxQ_\alpha - \int_\Omega \Psi_\beta(x)s(x)dx - \Psi_\beta(x)k(x)\frac{dT^N}{dx}\bigg|_{\partial\Omega} = 0 \qquad \text{for} \quad 1 \le \alpha, \beta \le N \tag{9}$$

Step 4
We will select global, piecewise continuous Lagrange interpolating polynomials as our trial functions. These functions can be linear, quadratic, cubic, or higher order - the choice is ours!

Step 5
Rather than dealing with N unique trial functions, we shall, with no loss of accuracy, discretize the problem domain into finite elements and utilize the resulting finite element basis functions.

$$\left(\Psi_\alpha(x)Q_\alpha\right)_e = \left\{N_k(\zeta_i)\right\}^T\left\{Q\right\}_e \tag{10a}$$

$$\left(\Psi_\beta(x)\right)_e = \left\{N_k(\zeta_i)\right\} \tag{10b}$$

The global distribution of the approximate solution can then be recovered by assembling the element level approximate solutions

$$\Psi_\alpha(x)Q_\alpha = \mathrm{S}_e\left(\left\{N_k(\zeta_i)\right\}^T\left\{Q\right\}_e\right) \tag{11}$$

Step 6
We will now substitute our discretized and assembled form of the approximate solution into (9) to obtain the discrete Galerkin weak statement

$$\begin{aligned} GWS^h &= S_e GWS_e^h \\ &= S_e\left(\int_{\Omega_e}\frac{d\left\{N_k\right\}}{dx}k(x)\frac{d\left\{N_k\right\}^T}{dx}d\bar{x}\left\{Q\right\}_e - \int_{\Omega_e}\left\{N_k\right\}s(x)d\bar{x} - k(x)\frac{dT^N}{dx}\left\{\delta_k\right\} = \left\{0_k\right\}\right) \end{aligned} \tag{12}$$

We must now select the degree of our basis function, define the element lengths and material properties, and perform the required differentiations and integrations. Let us now step back and recognize that every governing differential equation will lead to a form similar to (12). Rather than letting the required differentiations and integrations be unique for every problem, let us develop the *syntactical GWS*[h] which will, courtesy special Matlab routines, perform all required calculus and assembly to yield the desired terminal $Ax = b$ matrix statement.

Examining (12), two integrals are evident. Let us initially assume the thermal conductivity and source to be constant and then generalize for x-dependent material properties. The integrals are thus

$$k \int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\bar{x} \{Q\}_e \qquad s \int_{\Omega_e} \{N_k\} d\bar{x}$$

Both integrals follow a pattern - they consist of six components

(data)	$(\text{data})_e$	$\{\text{data}\}_e^T$	(metric)	$[\text{FE Matrix}]$	$\{\text{Q or data}\}_e$
scalar data	scalar data	vector data		finite element	vector unknown / data
constant	*averaged*	*interpolated*	*exponent*	*master element*	*interpolated*
on all Ω_e	on each Ω_e	over each Ω_e	on l_e	matrix	over each Ω_e

The master element matrix is an integral of bases and bases derivatives. It consists of four components, taking the form

$$[\text{Mndk}]$$

where

M denotes matrix and assumes the value of
- A for 1-d
- B for 2-d
- C for 3-d

n number of bases in the integral

d boolean differentiation index for each basis
- 0 = not differentiated
- 1 = differentiated

k order of basis function
- L for Linear
- Q for Quadratic
- C for Cubic

Therefore

$$k \int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\bar{x} \{Q\}_e$$

becomes

$$(k)(\ )_e \{\ \}_e^T (-1)[\text{A211L}]\{Q\}_e$$

and

$$s \int_{\Omega_e} \{N_k\} d\bar{x}$$

becomes

$$(s)(\ )_e \{\ \}_e^T (1)[\text{A10L}]\{\ \}_e$$

How do we handle material properties that are a function of x? Unless we have a clever plan, their x-dependency will muck up our integrals and hence really muck up our FE master elements!

Source term $s(x)$

Option A : Employ element averaged values of source. Knowing the functional form of $s(x)$ and the global x locations of our nodes, we can calculate the nodal values and then average them over each element. Since the values have been averaged, they are constant on each element, hence

$$\begin{aligned} \int_{\Omega_e} \{N_k\} s(x) d\bar{x} \quad &\Rightarrow \quad \bar{s}_e \int_{\Omega_e} \{N_k\} d\bar{x} \\ &= \quad (\,)(\bar{s})_e \,\{\,\}_e^T \,(1)\,[\text{A10L}]\,\{\,\}_e \end{aligned} \tag{13a}$$

Option B : Employ our finite element basis functions and interpolate the source over the element.

$$s(x) \approx s^N(x) = S_e\left(\{N_k\}^T \{S\}_e\right)$$

$$\begin{aligned} \int_{\Omega_e} \{N_k\} s(x) d\bar{x} \quad &\Rightarrow \quad \int_{\Omega_e} \{N_k\}\{N_k\}^T d\bar{x}\{S\}_e \\ &= \quad (\,)(\,)_e \,\{\,\}_e^T \,(1)\,[\text{A200L}]\,\{S\}_e \end{aligned} \tag{13b}$$

Thermal conductivity $k(x)$

Option A: Employ element averaged values of conductivity.

$$\begin{aligned} \int_{\Omega_e} \frac{d\{N_k\}}{dx} k(x) \frac{d\{N_k\}^T}{dx} d\bar{x}\{Q\}_e \quad &\Rightarrow \quad \bar{k}_e \int_{\Omega_e} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\bar{x}\{Q\}_e \\ &= \quad (\,)(\bar{k})_e \,\{\,\}_e^T \,(-1)\,[\text{A211L}]\,\{Q\}_e \end{aligned} \tag{14a}$$

Option B : Interpolate the conductivity over the element.

$$k(x) \approx k^N(x) = S_e\left(\{N_k\}^T \{K\}_e\right) = S_e\left(\{K\}_e^T \{N_k\}\right)$$

$$\begin{aligned} \int_{\Omega_e} \frac{d\{N_k\}}{dx} k(x) \frac{d\{N_k\}^T}{dx} d\bar{x}\{Q\}_e \quad &\Rightarrow \quad \{K\}_e^T \int_{\Omega_e} \{N_k\} \frac{d\{N_k\}}{dx} \frac{d\{N_k\}^T}{dx} d\bar{x}\{Q\}_e \\ &= \quad (\,)(\,)_e \,\{K\}_e^T \,(-1)\,[\text{A3011L}]\,\{Q\}_e \end{aligned} \tag{14b}$$

Having developed the syntactical GWS_e^h expressions for the integral terms, the final step is to identify and name the key terms in (12). Rearranging (12)

$$S_e\left(\int_{\Omega_e}\frac{d\{N_k\}}{dx}k(x)\frac{d\{N_k\}^T}{dx}d\bar{x}\{Q\}_e=\int_{\Omega_e}\{N_k\}s(x)d\bar{x}+k(x)\frac{dT^N}{dx}\{\delta_k\}\right)$$

we have, on the element level

$$\int_{\Omega_e}\frac{d\{N_k\}}{dx}k(x)\frac{d\{N_k\}^T}{dx}d\bar{x}\{Q\}_e\equiv[DIFF]_e\{Q\}_e \tag{15a}$$

$$\int_{\Omega_e}\{N_k\}s(x)d\bar{x}\equiv\{SRC\}_e \tag{15b}$$

$$k(x)\frac{dT^N}{dx}\{\delta_k\}\equiv\{BFLX\}_e \tag{15c}$$

Casting the element level GWS_e^h terms (15a-c) in the form $Ax=b$

$$[DIFF]_e\{Q\}_e=\{SRC\}_e+\{BFLX\}_e \tag{16a}$$

and assembling to form the global GWS^h

$$\begin{aligned}GWS^h&=S_eGWS_e^h\\&=S_e\left([DIFF]_e\{Q\}_e=\{SRC\}_e+\{BFLX\}_e\right)\end{aligned}$$

where all terms will be calculated and assembled using the syntactical forms (13) and (14) and a vector of zeros with the appropriate boundary conditions. Matlab will be used to do these operations. In addition, the script files will apply the Dirichlet data and then solve to obtain the unknown nodal solutions.

Homework:

1. Determine the master matrices [A211L] and [A200L].
2. Apply the recipe steps 1 through 6, i.e. equation (12) for the following ODE and BC set

$$\mathcal{L}(h)=\frac{d}{dx}\left(\frac{1}{Pa}\frac{dh(x)}{dx}\right)-uh(x)=0\qquad\Omega\in(0,L)$$

$$\ell(h)=\frac{dh(x)}{dx}=0\qquad\partial\Omega\in L$$

$$h=h_o\qquad\partial\Omega\in 0$$