Interpolation Theory

Satisfied that we can generate closed form solutions to simple differential equations, let's look at how we can approximate these solutions. Our motivation is that if we can approximate a given function, i.e. a known solution to a differential equation, we may gain some insight into how to generate approximate solutions which "kind of" satisfy a given differential equation.

There are numerous ways to approximate a function: fourier series, least squares fits, etc. Let us define a series of constants multiplied by functions to form our one-dimensional approximation:

$$Q(x) \approx Q^N(x) \equiv a_1 \Psi_1(x) + a_2 \Psi_2(x) + \cdots + a_N \Psi_N(x) \tag{1}$$

We can express (1) using compactly via summation notation as

$$Q^N(x) = \sum_{\beta=1}^{N} a_\beta \Psi_\beta(x) \tag{2a}$$

We can also employ a vector multiplication according to

$$Q^N(x) = \{a_1 \quad a_2 \quad \cdots \quad a_N\} \begin{Bmatrix} \Psi_1(x) \\ \Psi_2(x) \\ \vdots \\ \Psi_N(x) \end{Bmatrix} = a^T \Psi \tag{2b}$$

Note that (1) readily accommodates the exact solution for $N = 1$ if $a_1 = 1$ and $\Psi_1(x) = Q(x)$. The exact solution may also be replicated using several terms. Recalling the thermal slab solution (7)

$$\boxed{T(x) = \frac{sL^2}{2k}\left[1 - \left(\frac{x}{L}\right)^2\right] + \frac{qL}{k}\left[1 - \left(\frac{x}{L}\right)\right] + T_b}$$

we can define for $N = 3$

$$\begin{aligned} a_1 &= \frac{sL^2}{2k} & \Psi_1(x) &= 1 - \left(\frac{x}{L}\right)^2 \\ a_2 &= \frac{qL}{k} & \Psi_2(x) &= 1 - \frac{x}{L} \\ a_3 &= T_b & \Psi_3(x) &= 1 \end{aligned}$$

A major challenge will be to find or define a suitably general set of functions $\Psi_\beta(x), 1 \le \beta \le N$ such that (2a,b) generates a "good" approximation. As we develop the finite element theory, the computationally convenient and mathematically robust form for the *trial function set* $\Psi_\beta(x)$ of (2a,b) will emerge.

Let us now choose the form of the approximation to be a simple power series.

$$Q^N(x) = \sum_{\beta=0}^{N} a_\beta x^\beta = a_0 + \sum_{\beta=1}^{N} a_\beta x^\beta \tag{3}$$

The constant a_0 provides for a nonzero boundary condition, all $\Psi_\beta(x)$ possess a nonzero first derivative, and all products are square integrable on $(0, L)$ for reasonable N. These properties will become quite important as we develop the finite element theory. Unfortunately, (3) exhibits a fatal flaw. For large values of N, the power series expansion (3) becomes highly oscillatory between evaluation points. Thus, the simple high-ordered power series (3) is not generally used for interpolation. Instead, N is typically restricted to be less than 3 and a sum of the series is employed.

On the figure above, the function $Q(x)$ spans the domain $(0, L)$. The domain Ω has been discretized into five elements Ω^h. The exact solution is known at six points, called *knots* or *nodes*. Hence

$$Q_1 = Q(X_1), \quad Q_2 = Q(X_2), \quad Q_3 = Q(X_3), \quad Q_4 = Q(X_4)$$

For now we will neglect the boundaries - they will follow readily from our internal analysis. Since we have four constants $a_\beta = Q_\beta$, we will need to develop four trial functions $\Psi_\beta(x)$. Noting that if $\Psi_\beta(X_\beta) = 1$ and zero at every other node, we will have the *exact* values at every node with some error in between. Plotting the *linear* variation in the trial functions over the domain

Heed that the trial functions are piecewise continuous throughout the domain. Linear trial functions of this form are called pyramid, hat, or chateu functions. Examining node two, the functional form of $\Psi_2(x)$ is

$$\Psi_2 = \begin{cases} \dfrac{x - X_1}{X_2 - X_1} & X_1 \le x \le X_2 \\ \dfrac{X_3 - x}{X_3 - X_2} & X_2 \le x \le X_3 \\ 0 & x < X_1 \quad \text{or} \quad x > X_3 \end{cases} \tag{4}$$

Returning to the boundaries, identical functions will be used except that the pieces which fall outside the domain are discarded since $T^N(x)$ exists only on $(0, L)$.

If we examine the distribution of the trial functions over the domain, we make the remarkable discovery that they are the same on every element!

The tremendous advantage is that instead of N trial functions $\Psi_\beta(x)$ we now have two linear *finite element basis functions* $N_1(x)$ and $N_2(x)$.

$$N_1(x) = \frac{X_R - x}{X_R - X_L} \qquad N_2(x) = \frac{x - X_L}{X_R - X_L} \tag{5a}$$

If we look at each element individually and transform to a local coordinate system $\bar{x} = x - X_L$ with domain $(0, l_e)$,

$$N_1(\bar{x}) = 1 - \frac{\bar{x}}{l_e} \qquad N_2(\bar{x}) = \frac{\bar{x}}{l_e} \tag{5b}$$

Thus the element approximation of the function becomes

$$Q_e(\bar{x}) = N_1 Q_1^e + N_2 Q_2^e = \sum_{j=1}^{2} N_j Q_j^e \tag{6a}$$

Employing vector notation

$$Q_e(\bar{x}) = N_1 Q_L^e + N_2 Q_R^e = \{N_1 \; N_2\} \begin{Bmatrix} Q_L^e \\ Q_R^e \end{Bmatrix} = \{N\}^T \{Q\}_e \tag{6b}$$

If the entire domain is discretized into M elements, the complete function approximation is given as

$$Q^N(x) = \sum_{e=1}^{M} \sum_{j=1}^{2} N_j Q_j^e \tag{7a}$$

$$Q^N(x) = \sum_{e=1}^{M} \{N\}^T \{Q\}_e \tag{7b}$$

This summation will give rise to a process called *assembly* where the element-level approximations are combined to give the entire approximate function.

From the above figure, we may inquire "how can we minimize the error in our approximation?"

1. Use lots of elements
2. Use higher order basis functions

The typical approach is to use more linear elements. However, considerable insight can be gained by using higher order basis functions. We shall hold this insight for later but now take the time to develop the higher order basis functions.

Linear Basis Function

First, let us re-derive the linear basis functions to set the stage for quadratic and cubic functions. Given an element domain Ω_e spanning $(0, l_e)$ with nodal values Q_L^e, Q_R^e,

the general form of a linear function on the domain is

$$Q_e(\bar{x}) = a + b\left(\frac{\bar{x}}{l_e}\right) \tag{8}$$

where the local spatial coordinate $\bar{x}$ has been non-dimensionalized by l_e. Evaluating (8) at the left and right nodes to obtain the unknown coefficients a and b

$$Q_e(\bar{x}=0) = Q_L^e \quad \Rightarrow \quad Q_L^e = a \tag{9a}$$
$$Q_e(\bar{x}=l_e) = Q_R^e \quad \Rightarrow \quad Q_R^e = a+b \tag{9b}$$

Solving (9a,b) for the undetermined coefficients

$$a = Q_L^e \tag{10a}$$
$$b = Q_R^e - Q_L^e \tag{10b}$$

Substituting (10a,b) into (8)

$$Q_e(\bar{x}) = Q_L^e + \left(Q_R^e - Q_L^e\right)\left(\frac{\bar{x}}{l_e}\right) \tag{11}$$

Equation (11) contains the nodal values as common factors. Regrouping by the nodal values

$$Q_e(\bar{x}) = \left(1 - \frac{\bar{x}}{l_e}\right)Q_L^e + \left(\frac{\bar{x}}{l_e}\right)Q_R^e \tag{12}$$

The basis functions used to interpolate the nodal values can now be pulled directly from (12)

$$N_1(\bar{x}) = 1 - \frac{\bar{x}}{l_e} \qquad N_2(\bar{x}) = \frac{\bar{x}}{l_e}$$

which is exactly the result (5b). We shall push our terminology one step further and define two key monomials

$$\zeta_1(\bar{x}) \equiv 1 - \frac{\bar{x}}{l_e} \qquad \zeta_2(\bar{x}) \equiv \frac{\bar{x}}{l_e} \tag{13}$$

Note that the basis function (13) is *nodally consistent*. Evaluating ζ_1 at the location of Q_L^e yields a 1 while evaluating ζ_1 at the location of Q_R^e yields a 0. Similarly, evaluating ζ_2 at the location of Q_L^e yields a 0 while evaluating ζ_2 at the location of Q_R^e yields a 1.

The linear element level interpolation can be expressed with vector notation as

$$Q_e(\bar{x}) = \zeta_1 Q_L^e + \zeta_2 Q_R^e = \{\zeta_1 \ \zeta_2\} \begin{Bmatrix} Q_L \\ Q_R \end{Bmatrix}_e \tag{14}$$

To complete the generalization, we shall define the linear basis function as

$$\boxed{\{N_{k=1}(\zeta_i)\} = \begin{Bmatrix} \zeta_1 \\ \zeta_2 \end{Bmatrix}} \tag{15}$$

where the N means "basis function" and the k indicates the degree. Thus the element level linear basis approximation becomes

$$Q_e(\bar{x}) = \{N_{k=1}(\zeta_i)\}^T \{Q\}_e \tag{16}$$

Quadratic Basis Function

Following (16), which states the linear basis approximation, the quadratic form must be

$$Q_e(\bar{x}) = \{N_{k=2}(\zeta_i)\}^T \{Q\}_e \tag{17}$$

Our only requirement is to establish the functional form of $\{N_{k=2}(\zeta_i)\}$, which will involve polynomials in the ζ_i defined in (13). Since (17) must contain quadratic terms in $\bar{x}$, the general form is

$$Q_e(\bar{x}) = a + b\left(\frac{\bar{x}}{l_e}\right) + c\left(\frac{\bar{x}}{l_e}\right)^2 \tag{18}$$

Since three undetermined coefficients appear in (18), there must be three nodal values of Q defined for Ω_e. The logical choice is to select the midpoint of Ω_e as illustrated below.

If the mid-element value of Q is denoted Q_M, the evaluation of (18) at each node yields the three equations necessary to obtain the undetermined coefficients.

$$Q_e(\bar{x}=0)=Q_L^e \quad \Rightarrow \quad Q_L^e = a \tag{19a}$$

$$Q_e\left(\bar{x}=\frac{l_e}{2}\right)=Q_M^e \quad \Rightarrow \quad Q_M^e = a+\frac{b}{2}+\frac{c}{4} \tag{19b}$$

$$Q_e(\bar{x}=l_e)=Q_R^e \quad \Rightarrow \quad Q_R^e = a+b+c \tag{19c}$$

Solving (19a-c) for the undetermined coefficients is straightforward; substituting this solution into (18) then yields

$$Q_e(\bar{x})=Q_L^e+\left(-3Q_L^e+4Q_M^e-Q_R^e\right)\left(\frac{\bar{x}}{l_e}\right)+\left(2Q_L^e-4Q_M^e+2Q_R^e\right)\left(\frac{\bar{x}}{l_e}\right)^2 \tag{20}$$

Equation (20) contains $\{Q\}_e \equiv \{Q_L, Q_M, Q_R\}_e^T$ as the common factor. Rearranging (20) and then substituting $\zeta_1 = 1-\bar{x}/l_e$ and $\zeta_2 = \bar{x}/l_e$ yields

$$\begin{aligned} Q_e(\bar{x}) &= \left(\zeta_1(2\zeta_1-1)\right)Q_L^e+\left(4\zeta_1\zeta_2\right)Q_M^e+\left(\zeta_2(2\zeta_2-1)\right)Q_R^e \\ &= \{N_{k=2}(\zeta_i)\}^T\{Q\}_e \end{aligned} \tag{21}$$

where

$$\boxed{\{N_{k=2}(\zeta_i)\} \equiv \begin{Bmatrix} \zeta_1(2\zeta_1-1) \\ 4\zeta_1\zeta_2 \\ \zeta_2(2\zeta_{21}-1) \end{Bmatrix}} \tag{22}$$

Cubic Basis Function

By now you have hopefully realized that developing higher order basis functions is merely an algebraic exercise. Heed that the additional accuracy of increasing the basis degree k has an associated computational cost, hence there is "no free lunch" in finite element analysis. In many instances, switching from linear to quadratic basis is adequate for a dramatic improvement in accuracy. Ina few instances, even additional accuracy may be required. for completeness then, the cubic basis function should be created to cement the formulational structure in the reader's mind. The element approximation definition is

$$Q_e(\bar{x}) = \{N_{k=3}(\zeta_i)\}^T\{Q\}_e \tag{23}$$

Recalling (8) and (18), the direct algebraic expression is

$$Q_e(\bar{x}) = a+b\left(\frac{\bar{x}}{l_e}\right)+c\left(\frac{\bar{x}}{l_e}\right)^2+d\left(\frac{\bar{x}}{l_e}\right)^3 \tag{24}$$

Equation (24) now requires four nodal values of Q to constrain the undetermined coefficients. Choosing the 1/3 points, it is left as a homework exercise to verify that

$$\boxed{\{N_{k=3}(\zeta_i)\} \equiv \frac{9}{2}\begin{Bmatrix} \zeta_1\left(\zeta_2^2 - \zeta_2 + \frac{2}{9}\right) \\ \zeta_1\zeta_2(2-3\zeta_2) \\ \zeta_1\zeta_2(3\zeta_2 - 1) \\ \zeta_2\left(\zeta_2^2 - \zeta_2 + \frac{2}{9}\right) \end{Bmatrix}} \tag{25}$$

Homework

1. Starting with (24), complete the derivation of the cubic basis function (25). You may use Maple for this.
2. For an element domain of $(0, \pi/2)$, interpolate $Q(x) = \cos(x)$ using linear, quadratic, and cubic basis functions. Clearly report the functional forms of $Q_e(\bar{x})$ for each approximation. Include a graph showing the original function and the three approximations. Comment on accuracy. You may use Maple for this.