Differential Equations

Differential equations are used to describe the behavior of physical systems. Unfortunately, the equations describing complex systems are *unsolvable*, hence the need for *approximate solutions* and the *finite element method*. Before we begin developing the finite element methodology, we must first "rediscover" our forgotten knowledge of differential equations. We begin with some definitions and characterizations:

Ordinary Differential Equations (ODE) - an equation differentiated with respect to only one independent variable

$$\mathsf{L}(Q(t)) = A + BQ(t) + C\frac{dQ(t)}{dt} + D\frac{d^2Q(t)}{dt^2} = 0$$

Partial Differential Equation (PDE) - an equation differentiated with respect to more than one independent variable

$$\mathsf{L}(Q(t,x)) = A + BQ(t.x) + C\frac{\partial Q(t,x)}{\partial t} + D\frac{\partial Q(t,x)}{\partial x} + E\frac{\partial^2 Q(t,x)}{\partial t^2} + F\frac{\partial^2 Q(t,x)}{\partial x^2} + G\frac{\partial^2 Q(t,x)}{\partial t\,\partial x} = 0$$

The L denotes "differential operator" and will be used henceforth to represent the differential equations.

Linear Differential Equation - a differential equation where terms are only added (subtracted)

$$\mathsf{L}(Q(t,x)) = A + BQ(t.x) + C\frac{\partial Q(t,x)}{\partial t} + D\frac{\partial Q(t,x)}{\partial x} + E\frac{\partial^2 Q(t,x)}{\partial x^2} = 0$$

Non-linear Differential Equation - a differential equation where terms are multiplied (divided)

$$\mathsf{L}(Q(t,x)) = A + BQ(t.x) + C\frac{\partial Q(t,x)}{\partial t} + Q(t,x)\frac{\partial Q(t,x)}{\partial x} + D\left(\frac{\partial^2 Q(t,x)}{\partial x^2}\right)^2 = 0$$

First Order Differential Equation - a differential equation whose highest degree of differentiation is one

$$\mathsf{L}(Q(t,x)) = A + BQ(t.x) + C\frac{\partial Q(t,x)}{\partial t} + D\frac{\partial Q(t,x)}{\partial x} = 0$$

Second Order Differential Equation - a differential equation whose highest degree of differentiation is two

$$\mathsf{L}(Q(t,x)) = A + BQ(t.x) + C\frac{\partial Q(t,x)}{\partial t} + D\frac{\partial Q(t,x)}{\partial x} + E\frac{\partial^2 Q(t,x)}{\partial t^2} + F\frac{\partial^2 Q(t,x)}{\partial x^2} + G\frac{\partial^2 Q(t,x)}{\partial t\,\partial x} = 0$$

Higher Order Differential Equation - a differential equations whose highest degree of differentiation is greater than two

For this course, we will be looking at equations taking the general form of

$$\boxed{\mathsf{L}(Q(t,x)) = \frac{\partial Q(t,x)}{\partial t} + AQ(t,x)\frac{\partial Q(t,x)}{\partial x} + B\frac{\partial^2 Q(t,x)}{\partial x^2} + C = 0} \qquad (1)$$

We will employ various simplifications as we develop the methodology to handle this *second order, non-linear partial differential equation*.

Boundary and Initial Conditions

To solve differential equations, we need some type of boundary and initial conditions. Boundary conditions fall into three categories:

Dirichlet - a fixed value of one of the state variables

$$Q(x_L) = Q_L$$

Neumann - a fixed value of the derivative of one of the state variables

$$\frac{dQ}{d\hat{n}}(x_L) = -q(x_L)$$

Robin - a combination of Dirichlet and Neumann conditions

$$\frac{dQ}{d\hat{n}}(x_L) = -h(Q - Q_r)$$

Recall we need as many boundary conditions as the order of our equation and the number of state variables.

Let us now generate a closed form solution to a simplified version of (1) - the steady-state heat equation.

We wish to determine the temperature distribution due to conduction of heat through a thick slab as shown above. The slab is thermally loaded by a prescribed heat flux q, applied at the surface $x = a$, while the other surface at $x = b$ is held at the constant temperature of $T = T_b$. Further assume that defines the distribution of thermal conductivity $k(x)$ and the internal heat source $s(x)$ to be constant.

The governing differential equation is

$$\mathcal{L}(T) = -\frac{d}{dx}\left(k\frac{dT}{dx}\right) - s = 0 \qquad a < x < b \tag{2a}$$

with associated boundary conditions

$$\ell(T) = k\frac{dT}{dn} - q_{in} = 0 \qquad x = a \tag{2b}$$

$$T = T_b \qquad x = b \tag{2c}$$

Note that the heat flux definition has been rewritten as a differential equation. The lowercase script ℓ distinguishes the boundary condition ODE from the governing ODE.

To solve, we need only to integrate twice and apply the boundary conditions. Integrating once:

$$-k\frac{dT}{dx}-sx+C=0 \tag{3a}$$

Applying boundary condition (2b) taking care with the sign of the outward pointing normal

$$-k\frac{dT}{dx}=q_{in} \qquad \text{at} \quad x=a$$

$$\therefore q_{in}-sa+C=0 \quad \Rightarrow \quad C=-q_{in}+sa \tag{3b}$$

Substituting (3b) into (3a)

$$-k\frac{dT}{dx}-sx+(sa-q)=0 \tag{4}$$

Integrating again:

$$-kT-s\frac{x^2}{2}+(sa-q)x+C=0 \tag{5a}$$

Applying boundary condition (2c)

$$T=T_b \qquad \text{at} \qquad x=b$$

$$-kT_b-s\frac{b^2}{2}+(sa-q)b+C=0 \quad \Rightarrow \quad C=kT_b+s\frac{b^2}{2}-(sa-q)b \tag{5b}$$

Substituting (5b) into (5a) and solving for $T(x)$

$$\boxed{T(x)=\frac{sb^2}{2k}\left[1-\left(\frac{x}{b}\right)^2\right]+\frac{sab}{k}\left[1-\left(\frac{x}{b}\right)\right]+\frac{qb}{k}\left[1-\left(\frac{x}{b}\right)\right]+T_b} \tag{6}$$

Note that the solution is *independent of an x-axis shift*, hence if we redefine

$$L=b-a \qquad \text{and} \qquad a=0$$

the solution becomes

$$\boxed{T(x)=\frac{sL^2}{2k}\left[1-\left(\frac{x}{L}\right)^2\right]+\frac{qL}{k}\left[1-\left(\frac{x}{L}\right)\right]+T_b} \tag{7}$$

Homework:
For the x-axis shifted boundary conditions, re-solve (2a) and obtain (7). Then substitute the redefined boundaries into (6) and verify that the solution is indeed shift independent. DO NOT USE MAPLE!!!