ALE-01: Required Math Skills

Note: You may NOT use Maple

1) Solve the following ordinary differential equation, with *k*, *s* and *q* constant.

$$\frac{d}{dx}\left(kT(x)\right) + sx - q = 0 \qquad , \qquad T(x_L) = T_A$$

Integrating with respect to *x*:

$$\int\left(\frac{d}{dx}\left(kT(x)\right) + sx - q\right)dx = 0$$

$$kT(x) + s\frac{x^2}{2} - qx + c = 0$$

where *c* is the arbitrary constant of integration

Applying the boundary condition to solve for *c*:

$$kT_A + s\frac{x_L^2}{2} - qx_L + c = 0 \quad \Rightarrow \quad c = qx_L - s\frac{x_L^2}{2} - kT_A$$

Substituting and rearranging:

$$\boxed{T(x) = \frac{s}{2k}\left(x_L^2 - x^2\right) + \frac{q}{k}\left(x - x_L\right) + T_A}$$

2) Differentiate the following function N_{11} with respect to *x*. Be sure to clearly show the chain rule.

$$N_{11} = \zeta_1 \qquad \textit{where} \qquad \zeta_1 = 1 - \frac{\bar{x}}{l_e} \quad \textit{and} \quad \bar{x} = x - x_L$$

Applying the chain rule:

$$\frac{dN_{11}}{dx} = \frac{dN_{11}(\zeta_1)}{d\zeta_1}\frac{d\zeta_1(\bar{x})}{d\bar{x}}\frac{d\bar{x}(x)}{dx}$$

where:

$$\frac{dN_{11}}{d\zeta_1} = 1 \quad , \quad \frac{d\zeta_1}{d\bar{x}} = -\frac{1}{l_e} \quad , \quad \frac{d\bar{x}}{dx} = 1$$

Substituting and simplifying:

$$\boxed{\frac{dN_{11}}{dx} = -\frac{1}{l_e}}$$

3) Integrate the following expression by parts, where *k* and *s* are constant.

$$\int_0^L \Phi(x)\left[-\frac{d}{dx}\left(k\frac{dT^N(x)}{dx}\right)\right]dx$$

By definition, integration by parts is $\int_0^L u\,dv = -\int_0^L v\,du + uv\Big|_0^L$

Where for our equation: $u = \Phi \qquad dv = -\frac{d}{dx}\left(k\frac{dT^N}{x}\right)dx \quad \Rightarrow \quad du = \frac{d\Phi}{dx}dx \qquad v = -k\frac{dT^N}{dx}$

Substituting:

$$\boxed{\int_0^L \Phi(x)\left[-\frac{d}{dx}\left(k\frac{dT^N(x)}{dx}\right)\right]dx = \int_0^L \frac{d\Phi(x)}{dx}k\frac{dT^N(x)}{dx}dx - \Phi(x)k\frac{dT^N(x)}{dx}\Bigg|_0^L}$$

Note the sign change on the first term on the RHS!

4) Put the following matrix *assembly* in the form $\mathbf{A}x = b$. Then simplify for $l_{e,12} = l_{e,23} = l_e$, $k_{12} = k_{23} = k$, and $s_{12} = s_{23} = s$.

$$\left(\frac{k_{12}}{l_{e,12}}\begin{bmatrix}1 & -1 & 0\\ -1 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{Bmatrix}Q_1\\Q_2\\Q_3\end{Bmatrix} + \frac{k_{23}}{l_{e,23}}\begin{bmatrix}0 & 0 & 0\\ 0 & 1 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}Q_1\\Q_2\\Q_3\end{Bmatrix}\right) - \frac{1}{2}\left(\begin{Bmatrix}l_{e,12}s_{12}\\ l_{e,12}s_{12}\\ 0\end{Bmatrix} + \begin{Bmatrix}0\\ l_{e,23}s_{23}\\ l_{e,23}s_{23}\end{Bmatrix}\right) = \begin{Bmatrix}0\\0\\0\end{Bmatrix}$$

For variable *k*, *l*

For variable *k*, l_e, and *s*:

$$\boxed{\begin{bmatrix}\frac{k_{12}}{l_{e,12}} & -\frac{k_{12}}{l_{e,12}} & 0\\ -\frac{k_{12}}{l_{e,12}} & \frac{k_{12}}{l_{e,12}} + \frac{k_{23}}{l_{e,23}} & -\frac{k_{23}}{l_{e,23}}\\ 0 & -\frac{k_{23}}{l_{e,23}} & \frac{k_{23}}{l_{e,23}}\end{bmatrix}\begin{Bmatrix}Q_1\\Q_2\\Q_3\end{Bmatrix} = \begin{Bmatrix}\frac{l_{e,12}s_{12}}{2}\\ \frac{l_{e,12}s_{12}}{2} + \frac{l_{e,23}s_{23}}{2}\\ \frac{l_{e,23}s_{23}}{2}\end{Bmatrix}}$$

For constant *k*, l_e, and *s*:

$$\boxed{\frac{k}{l_e}\begin{bmatrix}1 & -1 & 0\\ -1 & 2 & -1\\ 0 & -1 & 1\end{bmatrix}\begin{Bmatrix}Q_1\\Q_2\\Q_3\end{Bmatrix} = \frac{s\,l_e}{2}\begin{Bmatrix}1\\2\\1\end{Bmatrix}}$$