Name:

ECE 597: Probability, Random Processes, and Estimation

Exam #1

Thursday April 2, 2015

1) Assume we have the joint density

$$f_{\mathbf{X},\mathbf{Y}}(x, y) \;=\; \frac{1}{3}(xy+1) \;\; 0 \le x \le 1,\; 0 \le y \le 2$$

Note: The ranges of $\mathbf{X}$ and $\mathbf{Y}$ are different
a) Determine the marginal density $f_{\mathbf{X}}(x)$
b) Determine the marginal density $f_{\mathbf{Y}}(y)$
c) Are $\mathbf{X}$ and $\mathbf{Y}$ independent? Why or why not?
d) Determine $E[\mathbf{X}|\mathbf{Y}]$ (note that this will be a function of y)

2) Assume we have an experiment where the random variable $\mathbf{X}$ is assumed to follow an Erlang density, i.e.,

$$f_{\mathbf{X}}(x) \quad = \quad \theta^2 x e^{-\theta x} \quad 0 < x < \infty$$

Assume we preform the experiment n times with outcomes $x_1, x_2, \ldots, x_n$, What is the maximum liklihood estimate of $\hat{\theta}$ based on these observations?

3) Assume $\underline{\mathbf{X}}$ and $\underline{\mathbf{Y}}$ are random vectors, not necessarily of the same size. Assume also that $K_{\mathbf{XX}}$, $K_{\mathbf{WW}}$ and $K_{\mathbf{WX}}$ are known. Now we make a new random vector

$$\underline{\mathbf{Y}} \;=\; A\underline{\mathbf{X}} + B\underline{\mathbf{W}} + \underline{C}$$

where A and B are constant matrices (not necessarily of the same size), and $\underline{C}$ is a constant vector. Determine an expression for $K_{\mathbf{YY}}$ in terms of these known quantities **ONLY**. Do **not** assume the means are zero.

Hint: $(FG)^T = G^T F^T$

The two formulas may (or may not) be useful in the following problem.

The general formula for a multidimensional Gaussian density is

$$f_{\underline{\mathbf{X}}}(\underline{x}) = \frac{1}{(2\pi)^{\frac{n}{2}}\left[det(K_{\mathbf{XX}})^{\frac{1}{2}}\right]} \; exp\; \left\{-\frac{1}{2}(\underline{x}-\underline{\mu})^T K_{\mathbf{XX}}^{-1}(\underline{x}-\underline{\mu})\right\}$$

The inverse of a 2 x 2 matrix is given as

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

4) Assume the random vector $\underline{\mathbf{X}} = [\mathbf{X}_1 \; \mathbf{X}_2]^T$ has the Gaussian density given by

$$f_{\underline{\mathbf{X}}}(\underline{x}) = \frac{1}{\pi\sqrt{3}} \; exp\left\{-\frac{1}{3}\left[2x_1^2 + 2x_1(x_2-1) + 2(x_2-1)^2\right]\right\}$$

Determine $\underline{\mu}$ and $K_{\mathbf{XX}}$

5) Assume $\underline{\mathbf{X}}$ is a zero mean random Gaussian vector with

$$K_{\mathbf{XX}} = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Determine a transformation matrix A and vector $\underline{b}$ such that

$$\underline{\mathbf{Y}} = A\underline{\mathbf{X}} + \underline{b}$$

and

$$K_{\mathbf{YY}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad \underline{\mu}_{\mathbf{Y}} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$