ECE-597: Optimal Control
Homework #2

Due Thursday September 20, 2007

1. (*From Chong and Zak*) Consider the problem:

$$\begin{array}{rl} \text{minimize} & x_1 x_2 - 2x_1 \\ \text{subject to} & x_1^2 - x_2^2 = 0 \end{array}$$

a) Show that if a solution exists ($H_x = 0$), it must be either $[1, 1]^T$ or $[-1, 1]^T$
b) Use the second oder sufficient conditions

$$\Delta x^T H_{xx} \Delta x > 0$$

for all Δx such that $f_x \Delta x = 0$ to show that $[1, 1]^T$ is the minimizer (and $[-1, 1]^T$ does not meet the sufficiency conditions).

2.(*From Chong and Zak*) Consider the problem of minimizing

$$\Pi = \frac{x^T Q x}{x^T P x}$$

where $Q = Q^T > 0$ and $P = P^T > 0$.
a) Show that if z is a solution, then so is az for all nonzero scalars a. Hence to avoid multiplicity of solutions, we impose the constraint

$$x^T P x = 1$$

b) Now solve the problem

$$\begin{array}{rl} \text{minimize} & x^T Q x \\ \text{subject to} & x^T P x - 1 = 0 \end{array}$$

In particular, show that the vectors x that solve this problem are eigenvectors that solve the following eigenproblem:

$$P^{-1} Q x = \lambda x$$

and then show that the maximum value of $x^T Q x$ is in fact equal to the largest eigenvalue of this eigenproblem.

3. (*From Chong and Zak*) Consider the sequence $\{x_k\}$ generated by

$$x_{k+1} = ax_k + bu_k$$

where a and b are real and nonzero. In particular, we want to find u_0 and u_1 such that x_2 is zero and the average input energy $\frac{1}{2}(u_0^2 + u_1^2)$ is minimized.
a) Find expressions for u_0 and u_1 in terms of a, b, x_0.
b) Look at the second order conditions to determine if you do, indeed, have a minimum.

Read the Appendix before doing the following three problems

4. Assume we have the optimization problem

$$\begin{aligned} \text{minimize } J &= x(N)^2 + \sum_{k=0}^{k=N-1} x(k)u(k) \\ \text{subject to} \quad & x(k+1) = x(k)u(k) + u(k)^2 \\ & x(k) \in \{-1, 0, 1, 2\}, \quad u(k) \in \{-1, 0, 1\}, \quad N = 2 \end{aligned}$$

a) Use the dynamic programming method (by hand) to determine the optimal cost to go and the optimal controls starting at any of our given states.

b) Copy *dynamic_example_b.m* and modify it to solve this problem. Your computer solution should be the same when you did it by hand (except when there is a choice of controls with equal cost). *Turn in your code and e-mail it to me.*

Notes: (1) You should get $J_0^*(-1) = -1, J_0^*(0) = -1, J_0^*(1) = -1, J_0^*(2) = -3$, *and (2) This is a strange dynamic programming problem since the cost is allowed to be negative. This is not used very often.*

5. Assume we have the optimization problem

$$\begin{aligned} \text{minimize } J &= (x(N) - 1)^2 + \sum_{k=0}^{k=N-1} \{x(k)^2 + \frac{1}{2}u(k)^2\} \\ \text{subject to} \quad & x(k+1) = x(k) + u(k) \\ & x(k) \in \{-2, -1, 0, 1, 2\}, \quad u(k) \in \{-1, 0, 1\}, \quad N = 2 \end{aligned}$$

a) Use the dynamic programming method (by hand) to determine the optimal cost to go and the optimal controls starting at any of our given states. *You should get* $J_0^*(-2) = 7, J_0^*(-1) = 2, J_0^*(0) = 0.5, J_0^*(1) = 2, J_0^*(2) = 5.5$

b) Copy *dynamic_example_b.m* and modify it to solve this problem. Your computer solution should be the same when you did it by hand (except when there is a choice of controls with equal cost).

c) Determine, by interpolation, the optimal controls and cost for initial conditions $x(0) = 0.5$ and $x(0) = 1.25$. Note that there is more than one optimal path here. Also, since we are using interpolation, the optimal cost may not be exactly the same for both paths.

d) Uncomment and modify the code at the end of the main routine to produce the optimal controls and cost for the same initial conditions as in part **c**. Turn in your code and your plots.

e) For this simple problem, we can write

$$J_0 = (x(2) - 1)^2 + x(0)^2 + \frac{1}{2}u(0)^2 + x(1)^2 + \frac{1}{2}u(1)^2$$

i) Show that if we substitute the state equations into the above formula to eliminate $x(1)$ and $x(2)$, and then minimize J_0 with respect to $u(0)$ and $u(1)$ we get the system of equations

$$\begin{aligned} 5u(0) + 2u(1) &= 2 - 4x(0) \\ 2u(0) + 3u(1) &= 2 - 2x(0) \end{aligned}$$

ii) Show that the optimal control is given by solving the equations

$$\begin{bmatrix} u(0) \\ u(1) \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 2 - 8x(0) \\ 6 - 2x(0) \end{bmatrix}$$

iii) Compare the exact results with your dynamic programming results for part **d**.

vi) Refine your mesh (the number of x and u) to verify that your dynamic programming solution is converging to the true optimal answer.

6. Assume we have the optimization problem

$$\begin{aligned} \text{minimize } J &= \sum_{k=0}^{k=N} \Big[|x(k) - 0.2k^2| + 0.25|u(k)|\Big] \\ \text{subject to } & \quad x(k+1) = x(k) + u(k) \\ & \quad x(k) \in \{0, 0.1, 0.2, 0.3, 0.4\}, \quad u(k) \in \{-0.2, -0.1, 0, 0.1, 0, 2\}, \quad N = 2 \end{aligned}$$

a) Use the dynamic programming method (by hand) to determine the optimal cost to go and the optimal controls starting at any of our given states. *You should get* $J_0^*(0) = 0.50$, $J_0^*(0.1) = 0.575$, $J_0^*(0.2) = 0.65$, $J_0^*(0.3) = 0.775$, $J_0^*(0.4) = 0.90$

b) Copy *dynamic_example_b.m* and modify it to solve this problem. Your computer solution should be the same when you did it by hand (except when there is a choice of controls with equal cost).

c) Determine, by interpolation, the optimal controls and cost for initial conditions $x(0) = 0.15$. Note that there may be more than one optimal path here.

d) Uncomment and modify the code at the end of the main routine to produce the optimal controls and cost for the same initial conditions as in part **c**. Turn in your code and your plots.

Appendix

Dynamic Programming programs

In class we did the following two simple examples using dynamic programming:

Example A. Assume we have the optimization problem

$$\begin{aligned} \text{minimize } J &= x(N)^2 + \sum_{k=0}^{k=N-1} u(k)^2 \\ \text{subject to } & \quad x(k+1) = x(k) + u(k) \\ & \quad x(k) \in \{0, 0.5, 1.0, 1.5\}, \quad u(k) \in \{-1, -0.5, 0, 0.5, 1\}, \quad N = 2 \end{aligned}$$

We can use the example program *dynamic_example_a.m* to solve this problem if we type:

```
dynamic_example_a(0,1.5,4,-1,1,5,0,2,3);
```

Compare the output of the program with our results from class.

Example B. Assume we have the optimization problem

$$\begin{aligned} \text{minimize } J &= 0.5x(N)^2 + \sum_{k=0}^{k=N-1} 0.25u(k)^2 \\ \text{subject to } & \quad x(k+1) = x(k) + 0.5u(k) \\ & \quad x(k) \in \{0, 0.5, 1.0\}, \quad u(k) \in \{-1, -0.5, 0, 0.5, 1\}, \quad N = 2 \end{aligned}$$

We can use the example program *dynamic_example_b.m* to solve this problem if we type:

```
dynamic_example_b(0,1,3,-1,1,5,0,2,3);
```

Compare the output of the program with our results from class.

You should try and figure out what is going on in the example code so you can figure out how to change it for the following problems.