<u>**ECE-521 Control Systems II**</u>

Homework 3

Due at the beginning of class, Tuesday January 4 , 2005

1) Consider the following state variable model

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

a) Consider transforming this to a new basis. Let $x = Qz$ where the columns of Q are the new basis vectors. Assuming Q^{-1} exists, show that in terms of this new basis we can write the state equations as

$$\dot{z} = [Q^{-1}AQ]z + [Q^{-1}B]u$$
$$y = [CQ]z + Du$$

b) Now consider the state variable model

$$\dot{x} = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$
$$y = [1 \quad 0]x$$

Compute the eigenvalues and eigenvectors of the A matrix, call the eigenvectors q_1 and q_2. Construct the matrix $Q = [q_1 \quad q_2]$.

c) Rewrite the system from part (b) using the eigenvectors as the basis vectors.

d) Determine an expression for A^2 in terms of A and I and then show explicitly that the matrix A satisfies its own characteristic equation by using the A matrix and evaluating both sides of the equation.

e) Using the Cayley-Hamilton method (matching on eigenvalues), show that

$$e^{At} = \begin{bmatrix} 2e^{2t} - e^{3t} & e^{2t} - e^{3t} \\ 2e^{3t} - 2e^{2t} & 2e^{3t} - e^{2t} \end{bmatrix}$$

2) For the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

a) Find the eigenvalues and characteristic equation for A.

b) Determine an expression for A^2 in terms of A and I and then show explicitly that the matrix A satisfies its own characteristic equation by using the A matrix and evaluating both sides of the equation.

c) Using the Cayley-Hamilton method (matching on eigenvalues), show that

$$e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

d) Compute e^{At} using the Laplace transform method.

3) Assume a matrix A is determined to have the following eigenvalues, $\lambda = -1, -1, -1, -2, -2, -3$. Determine the simultaneous equations that must be solved to determine e^{At} using the Cayley-Hamilton method. **DO NOT SOLVE!**

4) For

$$A = \begin{bmatrix} 2 & \sqrt{3} \\ \sqrt{3} & 4 \end{bmatrix}$$

determine $P(A)$ for $P(x) = x^4 - 5x^3 - x^2 + 6x + 1$ by computing the quotient $Q(x)$ and the remainder $R(x)$

$$P(x) = Q(x)\Delta(x) + R(x)$$

and then using $P(A) = R(A)$.

5) Controllability and observability are defined the same way for discrete time control systems as for continuous time control systems. However, sometimes it's easier to see what's happening in discrete time. Consider the discrete-time state variable system

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k$$

with initial state $x_0 = [0 \quad 0]^T$. For this system assume

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = [0]$$

a) Show that the characteristic equation for A is given by $\lambda^2 - \lambda - 1 = 0$ so that $A^2 = A + I$.

b) After one time step we have

$$x_2 = Ax_1 + Bu_1 = A[Ax_0 + Bu_0] + Bu_1 = [AB \quad B]\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

Using the results from part (a), show that

$$x_3 = [AB \quad B]\tilde{u}_3 \,,\ \tilde{u}_3 = [u_0 + u_1 \quad u_0 + u_2]^T$$

and

$$x_4 = [AB \quad B]\tilde{u}_4 \,,\ \tilde{u}_4 = [2u_0 + u_1 + u_2 \quad u_0 + u_1 + u_3]^T$$

It should be clear that as we continue forward, we will get an expression of the form

$$x_n = \begin{bmatrix} AB & B \end{bmatrix}\tilde{u}_n$$

c) Is it possible to find an input $\tilde{u}_n$ so that $x_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$? Is it possible to find and input $\tilde{u}_n$ so that $x_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$? If the answer to both of these is yes, then the system is controllable, and we can find and input to go from the origin to any state.

d) Compute the controllability matrix. Is the system controllable?

6) Consider the discrete-time state variable system

$$x_{k+1} = Ax_k + Bu_k$$
$$y_k = Cx_k + Du_k$$

with initial state $x_0 = [0 \quad 0]^T$. For this system assume

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = [0]$$

a) Show that the characteristic equation for A is given by $\lambda^2 - 2\lambda + 1 = 0$ so that $A^2 = 2A - I$.

b) After one time step we have

$$x_2 = Ax_1 + Bu_1 = A[Ax_0 + Bu_0] + Bu_1 = [AB \quad B]\begin{bmatrix} u_0 \\ u_1 \end{bmatrix}$$

Using the results from part (a), show that

$$x_3 = [AB \quad B]\tilde{u}_3 , \quad \tilde{u}_3 = [2u_0 + u_1 \quad -u_0 + u_2]^T$$

and

$$x_4 = [AB \quad B]\tilde{u}_4 , \ \tilde{u}_4 = [3u_0 + 2u_1 + u_2 \quad -2u_0 - u_1 + u_3]^T$$

It should be clear that as we continue forward, we will get an expression of the form

$$x_n = \begin{bmatrix} AB & B \end{bmatrix}\tilde{u}_n$$

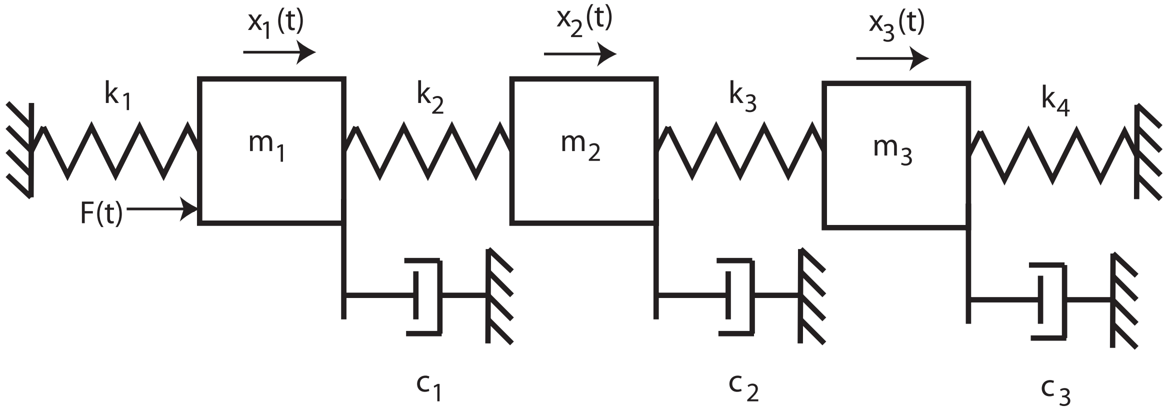
c) Is it possible to find an input $\tilde{u}_n$ so that $x_n = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$? Is it possible to find and input $\tilde{u}_n$ so that $x_n = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$? If the answer to both of these is yes, then the system is controllable, and we can find and input to go from the origin to any state.

d) Compute the controllability matrix. Is the system controllable?

Preparation for Lab 3 (To be done individually, No Maple):

In the following problem, only do what is asked of you. You don't need to derive everything...

7) Consider the following model of the three degree of freedom system we will be using in lab 3.



a) Using Lagrangian dynamics, show that the equations of motion can be written as

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1+k_2)x_1 &= F + k_2x_2 \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2+k_3)x_2 &= k_2x_1 + k_3x_3 \\ m_3\ddot{x}_3 + c_3\dot{x}_3 + (k_3+k_4)x_3 &= k_3x_2 \end{aligned}$$

b) Defining $q_1 = x_1$, $q_2 = \dot{x}_1$, $q_3 = x_2$, $q_4 = \dot{x}_2$, $q_5 = x_3$, and $q_6 = \dot{x}_3$, show that we get the following state equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \\ \dot{q}_5 \\ \dot{q}_6 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\left(\frac{k_1+k_2}{m_1}\right) & -\left(\frac{c_1}{m_1}\right) & \left(\frac{k_2}{m_1}\right) & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \left(\frac{k_2}{m_2}\right) & 0 & -\left(\frac{k_2+k_3}{m_2}\right) & -\left(\frac{c_2}{m_2}\right) & \left(\frac{k_3}{m_2}\right) & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \left(\frac{k_3}{m_3}\right) & 0 & -\left(\frac{k_3+k_4}{m_3}\right) & -\left(\frac{c_3}{m_3}\right) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{1}{m_1}\right) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} F$$

In order to get the A and B matrices for the state variable model, we need to determine all of the quantities in the above matrices. The C matrix will be determined by what we want the output of the system to be.

c) If we want the output to be the position of the second cart, what should C be? If we want the output to be the position of the first cart and the position of the third cart (two outputs), what should C be?

d) Now we will rewrite the equations from part (a) as

$$\begin{aligned}
\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 &= \frac{k_2}{m_1}x_2 + \frac{1}{m_1}F \\
\ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 &= \frac{k_2}{m_2}x_1 + \frac{k_3}{m_2}x_3 \\
\ddot{x}_3 + 2\zeta_3\omega_3\dot{x}_3 + \omega_3^2 x_3 &= \frac{k_3}{m_3}x_2
\end{aligned}$$

We will get our initial estimates of ζ_1, ω_1, ζ_2, ω_2, ζ_3, and ω_3 using the log-decrement method (assuming only one cart is free to move at a time). Assuming we have these parameters, show how $A_{2,1}$, $A_{2,2}$, $A_{4,3}$, $A_{4,4}$, $A_{6,5}$, and $A_{6,6}$ can be determined.

e) By taking the Laplace transforms of the above equations, we get the following (you don't need to do this)

$$\frac{X_3(s)}{F(s)} = \frac{\left(\frac{1}{m_1}\frac{k_2}{m_2}\frac{k_3}{m_3}\right)}{\Delta(s)}$$

where

$$\begin{aligned}
\Delta(s) \quad = \quad & (s^2 + 2\zeta_1\omega_1 s + \omega_1^2)(s^2 + 2\zeta_2\omega_2 s + \omega_2^2)(s^2 + 2\zeta_3\omega_3 s + \omega_3^2) \\
& -\left(\frac{k_2}{m_1}\right)\left(\frac{k_2}{m_2}\right)(s^2 + 2\zeta_3\omega_3 s + \omega_3^2) - \left(\frac{k_3}{m_2}\right)\left(\frac{k_3}{m_3}\right)(s^2 + 2\zeta_1\omega_1 s + \omega_1^2)
\end{aligned}$$

We would really rather write

$$\Delta(s) \quad = \quad (s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)(s^2 + 2\zeta_c\omega_c s + \omega_c^2)$$

Now we need to equate coefficients between powers of s in these two expressions for $\Delta(s)$. For s^6 we get $1 = 1$. For s^5 we get (note the symmetry!)

$$2\zeta_a\omega_a + 2\zeta_b\omega_b + 2\zeta_c\omega_c = 2\zeta_1\omega_1 + 2\zeta_2\omega_2 + 2\zeta_3\omega_3$$

For s^4 we get (note the symmetry!)

$$\begin{aligned}
&\omega_a^2 + \omega_b^2 + \omega_c^2 + 4\zeta_a\omega_a\zeta_b\omega_b + 4\zeta_a\omega_a\zeta_c\omega_c + 4\zeta_b\omega_b\zeta_c\omega_c \\
&= \omega_1^2 + \omega_2^2 + \omega_3^2 + 4\zeta_1\omega_1\zeta_2\omega_2 + 4\zeta_1\omega_1\zeta_3\omega_3 + 4\zeta_2\omega_2\zeta_3\omega_3
\end{aligned}$$

For s^3 we get (note the symmetry!)

$$\begin{aligned}
&\zeta_a\omega_a\omega_b^2 + \zeta_a\omega_a\omega_c^2 + \zeta_b\omega_b\omega_a^2 + \zeta_b\omega_b\omega_c^2 + \zeta_c\omega_c\omega_a^2 + \zeta_c\omega_c\omega_b^2 + 4\zeta_a\omega_a\zeta_b\omega_b\zeta_c\omega_c \\
&= \zeta_1\omega_1\omega_2^2 + \zeta_1\omega_1\omega_3^2 + \zeta_2\omega_2\omega_1^2 + \zeta_2\omega_2\omega_3^2 + \zeta_3\omega_3\omega_2^2 + \zeta_1\omega_1\omega_2^2 + 4\zeta_1\omega_1\zeta_2\omega_2\zeta_3\omega_3
\end{aligned}$$

These equations will be used to determine the final values of ζ_1 and ω_1. You need to write out the equations for the coefficients of s^2 (6 terms and 8 terms), s^1 (3 terms and 5 terms), and s^0 (1 term and 3 terms). *Note that there should be some symmetry in these!*

f) It is more convenient to write this as

$$\frac{X_3(s)}{F(s)} = \frac{K_3}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s+1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s+1)(\frac{1}{\omega_c^2}s^2 + \frac{2\zeta_c}{\omega_c}s+1)}$$

What is K_3 in terms of the k_i, m_i, and ω_a, ω_b, and ω_c?

g) Using the last equation in part (d) and the first transfer function in part (e), show that we can write

$$\frac{X_2(s)}{F(s)} = \frac{\frac{1}{m_1}\frac{k_2}{m_2}(s^2 + 2\zeta_3\omega_3 s + \omega_3^2)}{\Delta(s)}$$

h) This is more convenient to write as

$$\frac{X_2(s)}{F(s)} = \frac{K_2(\frac{1}{\omega_3^2}s^2 + \frac{2\zeta_3}{\omega_3}s+1)}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s+1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s+1)(\frac{1}{\omega_c^2}s^2 + \frac{2\zeta_c}{\omega_c}s+1)}$$

What is K_2 in terms of the k_i, m_i, and ω_a, ω_b, ω_c, and ω_3?

i) Show that

$$A_{6,3} = \frac{k_3}{m_3} = \frac{K_3}{K_2}\omega_3^2.$$

j) Using the second equation in part (d), the first transfer function in part (e), and the transfer function in part (g), show that we can write

$$\frac{X_1(s)}{F(s)} = \frac{\frac{1}{m_1}\left[(s^2 + 2\zeta_2\omega_2 s + \omega_2^2)(s^2 + 2\zeta_3\omega_3 s + \omega_3^2) - \frac{k_3}{m_2}\frac{k_3}{m_3}\right]}{\Delta(s)}$$

k) We would rather write this as

$$\frac{X_1(s)}{F(s)} = \frac{\frac{1}{m_1}\left[(s^2+2\zeta_x\omega_x s+\omega_x^2)(s^2+2\zeta_y\omega_y s+\omega_y^2)\right]}{\Delta(s)}$$

Again we need to equate the coefficients of equal powers of s. For s^4 we get $1=1$. For s^3 we get (note the symmetry!)

$$\zeta_x\omega_x+\zeta_y\omega_y=\zeta_2\omega_2+\zeta_3\omega_3$$

For s^2 we get (note the symmetry!)

$$\omega_x^2+\omega_y^2+4\zeta_x\omega_x\zeta_y\omega_y=\omega_2^2+\omega_3^2+4\zeta_2\omega_2\zeta_3\omega_3$$

For s^1 we get (note the symmetry!)

$$\zeta_x\omega_x\omega_y^2+\zeta_y\omega_y\omega_x^2=\zeta_2\omega_2\omega_3^2+\zeta_3\omega_3\omega_2^2$$

We will use these relationships to determine ζ_2 and ω_2. Verify that by equating the coefficients for s^0 we arrive at the relationship

$$A_{4,5}=\frac{k_3}{m_2}=\frac{\omega_2^2\omega_3^2-\omega_x^2\omega_y^2}{A_{6,3}}$$

l) A more convenient way to write this transfer function is

$$\frac{X_1(s)}{F(s)}=\frac{K_1\left[(\frac{1}{\omega_x^2}s^2+\frac{2\zeta_x}{\omega_x}s+1)(\frac{1}{\omega_y^2}s^2+\frac{2\zeta_y}{\omega_y}s+1)\right]}{(\frac{1}{\omega_a^2}s^2+\frac{2\zeta_a}{\omega_a}s+1)(\frac{1}{\omega_b^2}s^2+\frac{2\zeta_b}{\omega_b}s+1)(\frac{1}{\omega_c^2}s^2+\frac{2\zeta_c}{\omega_c}s+1)}$$

What is K_1 in terms of the k_i, m_i, and ω_a, ω_b, ω_c, ω_x, and ω_y?

m) Verify that

$$A_{4,1}=\frac{k_2}{m_2}=\frac{K_2}{K_1}\frac{\omega_x^2\omega_y^2}{\omega_3^2}$$

n) By using the results from part (e) for the s^0 term, verify that

$$A_{2,3}=\frac{k_2}{m_1}=\frac{\omega_1^2\omega_2^2\omega_3^2-\omega_a^2\omega_b^2\omega_c^2-A_{4,5}A_{6,3}\omega_1^2}{\omega_3^2A_{4,1}}$$

o) Verify that

$$B_2 = \frac{1}{m_1} = \frac{K_3\omega_a^2\omega_b^2\omega_c^2}{A_{6,3}A_{4,1}}$$

8) Modify either the one degree of freedom Simulink model (*Basic_1dof_State_Variable_Model.mdl*) and corresponding Matlab code, or the Simulink model and corresponding Matlab code from homework 2, to work with a three degree of freedom model. (The state model is one the class website.) Specifically, you need to

- Modify the matrix *get_desired_states* so that when you do the lab, the ECP system will output states x1, x1_dot, x2, x2_dot, x3, and x3_dot.
- Have 6 model outputs, m_x1, m_x1_dot, m_x2, m_x2_dot, m_x3, m_x3_dot
- Add plots of the positions and velocities of all three carts to the existing plots. All plots should be neatly organized on one page
- Using the state variable model on the web page, place the six closed loop poles at -10, -15, -20+10j, -20-10j, -40-20j, -40+20j
- By changing the C vector, control the position of the second cart so it follows a 1 cm step input
- By changing the C vector, control the position of the third cart, so it follows a 1 cm step input

You will need to turn in three plots, your Simulink code, and your Matlab code.

9) Utilizing the Simulink code from part 6, utilize the linear quadratic regulator method for choosing the state feedback gain K so that first cart follows a step input of amplitude 1 cm (we are trying to control the position of the first cart so that should be the system output). The control effort must remain below 0.4 and the settling time must be less than 0.5 seconds. Try also to avoid too many oscillations in carts 2 and 3. You need to turn in one plot and your Matlab code.