ECE 521: Control Systems II

Homework #1

Due: Tuesday March 16

For problems 1-5, let

$$\underline{a} = \begin{bmatrix} a \\ b \end{bmatrix}, \underline{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

and show the following:

1) for $f(\underline{x}) = \underline{a}^T \underline{x}$, $\frac{df}{d\underline{x}} = \underline{a}$

2) for $f(\underline{x}) = \underline{x}^T \underline{a}, \frac{df}{d\underline{x}} = \underline{a}$

3) for $f(\underline{x}) = A\underline{x}, \frac{df}{d\underline{x}} = A^T$

4) for $f(\underline{x}) = A^T \underline{x}, \frac{df}{d\underline{x}} = A$

5) for $f(\underline{x}) = \underline{x}^T A \underline{x}, \frac{df}{d\underline{x}} = (A + A^T)\underline{x}$

6) The error vector $\underline{e}$ between observation vector $\underline{d}$ and estimate of the input $\underline{\hat{x}}$ is $\underline{e} = \underline{d} - A\underline{\hat{x}}$. We want to weight the errors by a matrix R, where R is symmetric ($R = R^T$). Find $\underline{\hat{x}}$ to minimize $\underline{e}^T R \underline{e}$. (This is a weighted least squares.)

7) Show that any matrix A can be written as the sum of a symmetric matrix and a skew symmetric matrix. That is,

$$\begin{aligned} A &= R + Q \\ R &= R^T \\ Q &= -Q^T \end{aligned}$$

Determine R and Q.

8) Assume we expect a process to follow the following equation

$$y(t) = \frac{1}{ct + d\sqrt{t}}$$

Assume we measure the $y(t)$ at various times t:

t	$y(t)$
1.0	0.30
2.0	0.21
3.0	0.14
4.0	0.12
5.0	0.11
6.0	0.09

a) Determine a least squares estimate of the parameters c and d.
b) Estimate the value of $y(t)$ at $t = 2.5$.
c) Suppose we believe all measurements made before time $t = 3.5$ are twice as reliable as those made later. Determined a reasonable weighted least squares estimated of c and d.

9) Assume we expect a process to follow the following equation

$$\gamma(x) = \epsilon e^{\beta x}$$

Assume we measure the $\gamma(x)$ at various locations x:

x	$\gamma(x)$
0.0	2.45
0.1	2.38
0.4	2.30
2.0	1.40
4.0	0.70

a) Determine a least squares estimate of the parameters ϵ and β. (*Hint: Try logarythms...*)
b) Estimate the value of $\gamma(x)$ at $x = 3.0$.

10) Assume we have an experimental process we are modeling and, based on sound physical principles, we assume a relationship between x and y to be

$$y(x) = \left(\frac{\alpha}{x}\right)^{\beta}$$

and we have the following measurements

y	x
8	1
1	2
0.3	3
0.1	4

write out and describe how to use a least squares technique to estimate the parameters α and β. Assume we have the general form

$$\mathbf{a} = \mathrm{B}\mathbf{c}$$

What are in the **a** and **c** vectors? What is in the B matrix? What are your estimates for α and β?

Hint: you cannot solve for α directly. Let $w = \beta \log \alpha$ and solve for w and β, then infer α.

11) Linearize the following two systems about the origin, and write the results in state variable form

$$\begin{aligned} \dot{x}_1 &= x_1 x_2 + 3x_2 + u_1^2 + u_2 \\ \dot{x}_2 &= 4x_1 + x_2 + x_1 u_2^2 + 2u_1 \end{aligned}$$

$$\begin{aligned} \dot{x}_1 &= x_1^2 - \sin(3x_2) + u_1^3 - u_2 \\ \dot{x}_2 &= x_2 - u_1 + x_1 e^{-x_2} \end{aligned}$$

$\boxed{6}$ $\hat{\underline{x}} = \left(A^T R A\right)^{-1} A^T R \underline{d}$

$\boxed{8}$ $\hat{y}(t) = (0.8104t + 2.4707\sqrt{t})^{-1}$

$\boxed{9}$ $\hat{\gamma}(x) = 2.519e^{-0.3142x}$

$\boxed{10}$ $\hat{y}(x) = \left(\frac{1.974}{x}\right)^{3.1183}$

$\boxed{11}$

$$\begin{aligned} \delta\dot{\underline{x}} &= \begin{bmatrix} 0 & 3 \\ 4 & 1 \end{bmatrix} \delta\underline{x} + \begin{bmatrix} 0 & 1 \\ 2 & 0 \end{bmatrix} \delta\underline{u} \\ \delta\dot{\underline{x}} &= \begin{bmatrix} 0 & -3 \\ 1 & 1 \end{bmatrix} \delta\underline{x} + \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \delta\underline{u} \end{aligned}$$