ECE-520: Discrete-Time Control Systems
Homework 4

Due: Tuesday January 13 in class

1) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G(T)\underline{x}(k) + H(T)u(k)$$

where the explicit dependence of G and H on the sampling time T has been emphasized. Here

$$G(T) = e^{AT}$$
$$H(T) = \int_0^T e^{A\lambda} d\lambda B$$

a) Show that if A is invertible, we can write $H(T) = [e^{AT} - I]A^{-1}B$

b) Show that if A is invertible and T is small we can write the state model as

$$\underline{x}(k+1) = [I + AT]\underline{x}(k) + BTu(k)$$

c) Show that if we use the approximation

$$\dot{\underline{x}}(t) \approx \frac{\underline{x}((k+1)T) - \underline{x}(kT)}{T} = Ax(kT) + Bu(kT)$$

we get the same answer as in part **b,** but using this approximation we do not need to assume A is invertible.

d) Show that if we use two terms in the approximation for e^{AT} (and no assumptions about A being invertible), we can write the state equations as

$$\underline{x}(k+1) = \left[I + AT\right]\underline{x}(k) + [T + \frac{1}{2}AT^2]Bu(k)$$

2) For the state variable system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t)$$

a) Show that

$$e^{AT} = \begin{bmatrix} 2e^{2T} - e^{3T} & e^{2T} - e^{3T} \\ 2e^{3T} - 2e^{2T} & 2e^{3T} - e^{2T} \end{bmatrix}$$

b) Derive the equivalent ZOH discrete-time system

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

for $T = 0.1$ (integrate each entry in the matrix $e^{A\lambda}$ separately.) Compare your answer with that given by Matlab's **c2d** command.

3) For the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

a) Find the eigenvalues and characteristic equation for A.

b) Determine an expression for A^2 in terms of A and I and then show explicitly that the matrix A satisfies its own characteristic equation by using the A matrix and evaluating both sides of the equation.

c) Using the Cayley-Hamilton method (matching on eigenvalues), show that

$$e^{At} = \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

d) Compute e^{At} using the Laplace transform method.

4) For the continuous time model

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix} \underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t-0.03)$$
$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \underline{x}(t)$$

derive the equivalent ZOH (zero order hold, this is our standard method of sampling) discrete-time system

$$\begin{bmatrix} \underline{x}([k+1]T) \\ u(kT) \end{bmatrix} = \begin{bmatrix} G & H_1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{x}(kT) \\ u([k-1]T) \end{bmatrix} + \begin{bmatrix} H_0 \\ I \end{bmatrix} u(kT)$$
$$y(kT) = C \begin{bmatrix} \underline{x}(kT) \\ u([k-1]T) \end{bmatrix}$$

for $T = 0.1$. Specifically, determine G, H_0, H_1, and C. You should do all of the calculations by hand (you've done most of the work in problem 2). You can check your answers in Matlab using the **c2d** command and the **expm** command. Assume we want the system output to remain the same.

5) Sometimes we would like to know what is happening to our continuous time system

$$\underline{x}(t) = A\underline{x}(t) + B\underline{u}(t)$$
$$\underline{y}(t) = C\underline{x}(t) + D\underline{u}(t)$$

between sample times, such as at time $t = kT + \Delta T$ where ΔT is less than the sampling interval T. From class, the solution to the continuous-time state equation system is given by

$$\underline{x}(t) = e^{A(t-t_o)}\underline{x}(t_o) + \int_{t_o}^{t} e^{A(t-\lambda)}B\underline{u}(\lambda)d\lambda$$

Assuming $t_o = kT$ and $t = kT + \Delta T$, derive an expression in terms of $x(kT)$ and $u(kT)$ for the output at time $t = kT + \Delta T$, i.e., find $\underline{y}(kT + \Delta T)$. Do not assume D is zero.

6) (A sure sign of the apocalypse…you are to use Maple for this entire problem!)

a) Show that the state variable model

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$
$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t)$$

corresponds to the transfer function

$$G_p(s) = \frac{1}{(s+1)(s+2)}$$

b) Compute *G* and *H* for this problem, assuming a sample interval *T*.

c) Use the discrete-time state variable equations to determine the corresponding transfer function. Note that we previously determined that the corresponding discrete-time transfer function for this plant (assuming a zero order hold) is

$$G_p(z) = \frac{z(0.5 - e^{-T} + 0.5e^{-2T}) + (0.5e^{-T} - e^{-2T} + 0.5e^{-3T})}{(z - e^{-T})(z - e^{-2T})}$$

It helps if you tell Maple to assume *T* is real.