ECE-520: Discrete-Time Control Systems
Homework 2

Due: Tuesday December 16 at the beginning of class

1) For the z -transform

$$X(z) = \frac{3}{z-2}$$

a) Show that, by multiplying and dividing by z and then using partial fractions, the corresponding discrete-time sequence is

$$x(k) = -\frac{3}{2}\delta(k) + \frac{3}{2}2^k u(k)$$

b) By starting with the z -transform

$$Y(z) = \frac{3z}{z-2}$$

and the z -transform properties, show that

$$x(k) = 3\,2^{k-1}u(k-1)$$

2) For impulse response $h(n) = \left(\frac{1}{2}\right)^{n-3} u(n-1)$ and input $x(n) = \left(\frac{1}{4}\right)^{n+1} u(n-2)$, use z-transforms of the input and impulse response to show the system output is

$$y(n) = \left[\left(\frac{1}{2}\right)^{n} - \left(\frac{1}{4}\right)^{n-1}\right]u(n-3)$$

3) For impulse response $h(n) = \left(\frac{1}{3}\right)^{n-1} u(n-2)$ and input $x(n) = (n-1)\left(\frac{1}{2}\right)^{n-2} u(n-3)$, use z-transforms to show that the system output is

$$y(n) = \left[\left(\frac{1}{3}\right)^{n-4} + \left(\frac{1}{2}\right)^{n-4} (n-5)\right]u(n-5)$$

4) Consider the following difference equation

$$x(k+2)-4x(k+1)+4x(k)= f(k)$$

Assume all initial conditions are zero.

a) Determine the *impulse response* of the system, i.e., the response $x(k)$ when $f(k)=\delta(k)$.

b) Determine $x(0), x(1), x(2), x(3),$ and $x(4)$ from your answer to **a.** Compare this answer with the known values of $x(0)$ and $x(1)$. Using the difference equation compute $x(3)$ and $x(4)$ and compare these values to those in your solution.

c) Determine the *step response* of the system, i.e., the response $x(k)$ when $f(k)=u(k)$

d) Determine $x(0), x(1), x(2), x(3),$ and $x(4)$ from your answer to **c.** Compare this answer with the known values of $x(0)$ and $x(1)$. Using the difference equation compute $x(3)$ and $x(4)$ and compare these values to those in your solution.

5) Consider the difference equation

$$x(k+2)-5x(k+1)+6x(k)= f(k)$$

where $f(k)=u(k)$, a unit step. Assume $x(0)=1$ and $x(1)=1$.

a) Determine the *Zero Input Response* (ZIR), $x_{ZIR}(k)$. This is the part of the solution $x(k)$ due to the initial conditions alone (assume the input is zero).

b) Determine the *Zero State Response* (ZSR), $x_{ZSR}(k)$. This is the part of the solution $x(k)$ due to the input alone (assume all initial conditions are zero).

c) Find the total response $x(k)=x_{ZIR}(k)+x_{ZSR}(k)$

d) Find the transfer function and the impulse response.

e) Determine $x(0), x(1), x(2), x(3),$ and $x(4)$ from your answer to **c.** Compare this answer with the known values of $x(0)$ and $x(1)$. Using the difference equation compute $x(3)$ and $x(4)$ and compare these values to those in your solution.

6) In this problem we will find a really ugly z-transform pair and then use it. Unfortunately this form comes up a lot in real problems.

a) Use Euler's identity , $\cos(\alpha) = \frac{1}{2}\left[e^{j\alpha} + e^{-j\alpha}\right]$, to show that the z-transform of $g(n) = r\gamma^n \cos(\beta n + \theta)u(n)$ is given by $G(z) = \frac{rz[z\cos(\theta) - \gamma\cos(\beta - \theta)]}{z^2 - 2\gamma\cos(\beta)z + \gamma^2}$

In what follows we will write

$$G(z) = \frac{rz[z\cos(\theta) - \gamma\cos(\beta - \theta)]}{z^2 - 2\gamma\cos(\beta)z + \gamma^2} = \frac{Az^2 + Bz}{z^2 + 2az + \gamma^2}$$

b) Show that $\cos(\beta) = \frac{-a}{\gamma}$

c) Using the identity trigonometric identity $\cos(\beta - \theta) = \cos(\beta)\cos(\theta) + \sin(\beta)\sin(\theta)$ show that $\theta = \tan^{-1}\left(\frac{aA - B}{A\sqrt{\gamma^2 - a^2}}\right)$

d) Using the trigonometric identity $\cos^2(\theta) + \sin^2(\theta) = 1$, show that

$$r = \sqrt{\frac{A^2\gamma^2 + B^2 - 2AaB}{\gamma^2 - a^2}}$$

Now we know how to find all of the parameters!

e) Us the above formula to find the impulse response $g(n)$ for $G(z) = \frac{z^2 + 0.5z}{z^2 + 0.2z + 0.125}$.

f) Compute $g(n)$ from part **e** for $n = 0,1,2,3,4$ and then perform long division to verify that your answer to **e** is correct for these terms

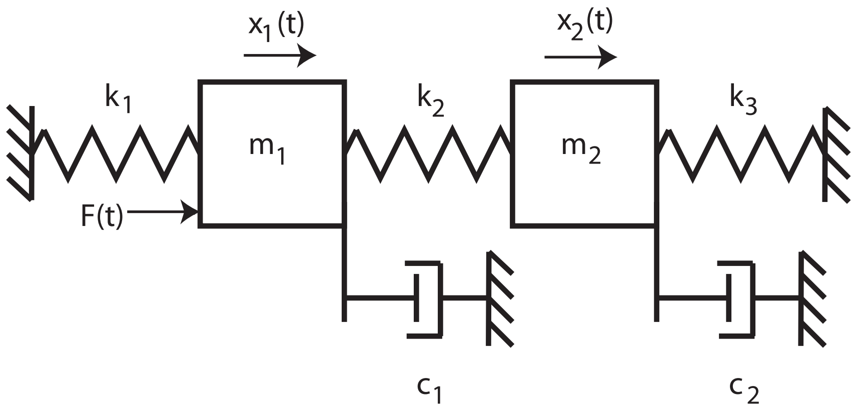
g) Determine the unit step response $y(n)$ for $G(z) = \frac{1}{z^2 + 0.1z + 4}$, by looking using the form $\frac{Y(z)}{z} = \frac{1}{(z-1)(z^2 + 0.1z + 4)} = \frac{\alpha_1}{z-1} + \frac{\alpha_2 z + \alpha_3}{z^2 + 0.1z + 4}$

Hint: α_1 can be found using the cover-up method, α_2 can be found by multiplying both sides by z and letting $z \to \infty$, and α_3 can be found by substituting a convenient value for z, like $z = 0$.

h) Compute $y(n)$ from part **e** for $n = 0,1,2,3,4$ and then perform long division to verify that your answer to **e** is correct for these terms

Preparation for Lab 2 (to be done individually, No Maple)

7) Consider the following model of the two degree of freedom system we will be using in lab 2.



a) Draw free body diagrams for each mass and show that the equations of motion can be written as

$$\begin{aligned} m_1\ddot{x}_1 + c_1\dot{x}_1 + (k_1 + k_2)x_1 &= F + k_2x_2 \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + (k_2 + k_3)x_2 &= k_2x_1 \end{aligned}$$

b) Defining $q_1 = x_1$, $q_2 = \dot{x}_1$, $q_3 = x_2$, and $q_4 = \dot{x}_2$, show that we get the following state equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{k_1+k_2}{m_1}\right) & -\left(\frac{c_1}{m_1}\right) & \left(\frac{k_2}{m_1}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\frac{k_2}{m_2}\right) & 0 & -\left(\frac{k_2+k_3}{m_2}\right) & -\left(\frac{c_2}{m_2}\right) \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{1}{m_1}\right) \\ 0 \\ 0 \end{bmatrix} F$$

In order to get the A and B matrices for the state variable model, we need to determine all of the quantities in the above matrices. The C matrix will be determined by what we want the output of the system to be.

c) If we want the output to be the position of the first cart, what should C be? If we want the output to be the position of the second cart what should C be?

d) Now we will rewrite the equations from part (a) as

$$\ddot{x}_1 + 2\zeta_1\omega_1\dot{x}_1 + \omega_1^2 x_1 = \frac{k_2}{m_1}x_2 + \frac{1}{m_1}F$$
$$\ddot{x}_2 + 2\zeta_2\omega_2\dot{x}_2 + \omega_2^2 x_2 = \frac{k_2}{m_2}x_1$$

We will get our initial estimates of ζ_1, ω_1, ζ_2, and ω_2 using the log-decrement method (assuming only one cart is free to move at a time). Assuming we have measured these parameters, show how $A_{2,1}$, $A_{2,2}$, $A_{4,3}$, and $A_{4,4}$ can be determined.

e) By taking the Laplace transforms of the equations from part (d), show that we get the following transfer function

$$\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1m_2}\right)}{(s^2+2\zeta_1\omega_1 s+\omega_1^2)(s^2+2\zeta_2\omega_2 s+\omega_2^2)-\frac{k_2^2}{m_1m_2}}$$

f) It is more convenient to write this as

$$\frac{X_2(s)}{F(s)} = \frac{\left(\frac{k_2}{m_1m_2}\right)}{(s^2+2\zeta_a\omega_a s+\omega_a^2)(s^2+2\zeta_b\omega_b s+\omega_b^2)}$$

By equating powers of s in the denominator of the transfer function from part (e) and this expression you should be able to write down four equations. The equations corresponding to the coefficients of s^3, s^2, and s do not seem to give us any new information, but they will be used to get consistent estimates of ζ_1 and ω_1. The equation for the coefficient of s^0 will give us a new relationship for $\frac{k_2^2}{m_1m_2}$ in terms of the parameters we will be measuring.

g) We will actually be fitting the frequency response data to the following transfer function

$$\frac{X_2(s)}{F(s)} = \frac{K_2}{(\frac{1}{\omega_a^2}s^2+\frac{2\zeta_a}{\omega_a}s+1)(\frac{1}{\omega_b^2}s^2+\frac{2\zeta_b}{\omega_b}s+1)}$$

What is K_2 in terms of the parameters of part (f)?

h) Using the transfer function in (f) and the Laplace transform of the second equation in part (d), show that the transfer function between the input and the position of the first cart is given as

$$\frac{X_1(s)}{F(s)} = \frac{\frac{1}{m_1}\left(s^2 + 2\zeta_2\omega_2 s + \omega_2^2\right)}{(s^2 + 2\zeta_a\omega_a s + \omega_a^2)(s^2 + 2\zeta_b\omega_b s + \omega_b^2)}$$

i) This equation is more convenient to write in the form

$$\frac{X_1(s)}{F(s)} = \frac{K_1\left(\frac{1}{\omega_2^2}s^2 + \frac{2\zeta_2}{\omega_2}s + 1\right)}{(\frac{1}{\omega_a^2}s^2 + \frac{2\zeta_a}{\omega_a}s + 1)(\frac{1}{\omega_b^2}s^2 + \frac{2\zeta_b}{\omega_b}s + 1)}$$

What is K_1 in terms of the quantities given in part (h)?

j) Verify that $A_{4,1} = \frac{k_2}{m_2} = \frac{K_2}{K_1}\omega_2^2$

k) Verify that $A_{2,3} = \frac{k_2}{m_1} = \frac{\omega_1^2\omega_2^2 - \omega_a^2\omega_b^2}{A_{4,1}}$

l) Verify that $B_2 = \frac{1}{m_1} = \frac{K_2\omega_a^2\omega_b^2}{A_{4,1}}$. Note that this term contains all of the scaling and unit conversions.