ECE-320: Linear Control Systems
Homework 8

Due: Thursday February 7 at the beginning of class

1) Consider the discrete-time state variable model $\underline{x}(k+1) = G(T)\underline{x}(k) + H(T)u(k)$

where the explicit dependence of G and H on the sampling time T has been emphasized. Here

$$G(T) = e^{AT}$$

$$H(T) = \int_0^T e^{A\lambda} d\lambda B$$

a) Show that if A is invertible, we can write $H(T) = [e^{AT} - I]A^{-1}B$

b) Show that if A is invertible and T is small we can write the state model as

$$\underline{x}(k+1) = [I + AT]\underline{x}(k) + BTu(k)$$

c) Show that if we use the approximation

$$\dot{\underline{x}}(t) \approx \frac{\underline{x}((k+1)T) - \underline{x}(kT)}{T} = Ax(kT) + Bu(kT)$$

we get the same answer as in part **b,** but using this approximation we do not need to assume A is invertible.

d) Show that if we use two terms in the approximation for e^{AT} (and no assumptions about A being invertible), we can write the state equations as

$$\underline{x}(k+1) = \left[I + AT\right]\underline{x}(k) + [IT + \frac{1}{2}AT^2]Bu(k)$$

2) For the state variable system

$$\dot{\underline{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}\underline{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}u(t)$$

a) Show that

$$e^{AT} = \begin{bmatrix} 2e^{2T} - e^{3T} & e^{2T} - e^{3T} \\ 2e^{3T} - 2e^{2T} & 2e^{3T} - e^{2T} \end{bmatrix}$$

b) Derive the equivalent ZOH discrete-time system

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

for $T = 0.1$ (integrate each entry in the matrix $e^{A\lambda}$ separately.) Compare your answer with that given by Matlab's **c2d** command, [G,H] = c2d(A,B,T).

3) For the matrix $A=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}$ show that $e^{At}=\begin{bmatrix}e^t & 0\\ te^t & e^t\end{bmatrix}$

4) Consider the discrete-time state variable model $\underline{x}(k+1)=G\underline{x}(k)+Hu(k)$ with the initial state $x(0)=0$. Let

$$G=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, H=\begin{bmatrix}1\\ 1\end{bmatrix}, C=\begin{bmatrix}1 & 0\end{bmatrix}, D=0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with $u(k)=G_{pf}r(k)-Kx(k)$ show that the transfer function is given by

$$F(z)=\frac{Y(z)}{R(z)}=C(zI-\tilde{G})^{-1}\tilde{H}=\frac{G_{pf}(z+1)}{(z+k_1)(z+k_2)-(k_1-1)(k_2-1)}$$

c) Show that if $G_{pf}=1$ and $k_1=k_2=0$, the transfer function reduces to that found in part **a.**

d) Is the system controllable? That is, is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero?

5) Consider the discrete-time state variable model

$$\underline{x}(k+1)=G\underline{x}(k)+Hu(k)$$

with the initial state $x(0)=0$. Let

$$G=\begin{bmatrix}1 & 0\\ 1 & 1\end{bmatrix}, H=\begin{bmatrix}0\\ 1\end{bmatrix}, C=\begin{bmatrix}0 & 1\end{bmatrix}, D=0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with $u(k)=G_{pf}r(k)-Kx(k)$ show that the transfer function is given by

$$F(z)=\frac{Y(z)}{R(z)}=\frac{G_{pf}(z-1)}{(z-1)(z+k_2-1)}$$

c) Show that if $G_{pf}=1$ and $k_1=k_2=0$, the transfer function reduces to that found in part **a.**

d) Is the system controllable?

6) Consider the discrete-time state variable model

$$\underline{x}(k+1) = G\underline{x}(k) + Hu(k)$$

with the initial state $x(0) = 0$. Let

$$G = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, H = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, C = \begin{bmatrix} 1 & 0 \end{bmatrix}, D = 0$$

a) Determine the corresponding transfer function for the system.

b) Using state variable feedback with $u(k) = G_{pf} r(k) - Kx(k)$ show that transfer function is given by

$$F(z) = \frac{Y(z)}{R(z)} = \frac{G_{pf}}{z^2 + (k_2 - 1)z + (k_1 - 1)}$$

c) Show that if $G_{pf} = 1$ and $k_1 = k_2 = 0$, the transfer function reduces to that found in part **a.**

d) Is it possible to find k_1 and k_2 to place the poles of the closed loop system where ever we want? For example, can we make both poles be zero? If we want the poles to be at p_1 and p_2 show that $k_2 = 1 - (p_1 + p_2)$ and $k_1 = 1 + p_1 p_2$.