

**ECE-320: Linear Control Systems**  
Homework 5

Due: Tuesday January 15 at the beginning of class

1) For the plant

$$G_p(s) = \frac{K}{\frac{1}{\omega_n^2} s^2 + \frac{2\zeta}{\omega_n} s + 1}$$

a) If the plant input is  $u(t)$  and the output is  $x(t)$ , show that we can represent this system with the differential equation

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2x(t) = K\omega_n^2u(t)$$

b) Assuming we use states  $q_1(t) = x(t)$  and  $q_2(t) = \dot{x}(t)$ , and the output is  $x(t)$ , show that we can write the state variable description of the system as

$$\frac{d}{dt} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

or

$$\dot{q}(t) = Aq(t) + Bu(t) \quad y(t) = Cq(t) + Du(t)$$

Determine the A, B, C and D matrices.

c) Assume we use state variable feedback of the form  $u(t) = G_{pf}r(t) - kq(t)$ , where  $r(t)$  is the new input to the system,  $G_{pf}$  is a prefilter (for controlling the steady state error), and  $k$  is the state variable feedback gain vector. Show that the state variable model for the closed loop system is

$$\dot{q}(t) = (A - Bk)q(t) + (BG_{pf})r(t)$$

$$y(t) = (C - Dk)q(t) + (DG_{pf})r(t)$$

or

$$\dot{q}(t) = \tilde{A}q(t) + \tilde{B}r(t)$$

$$y(t) = \tilde{C}q(t) + \tilde{D}r(t)$$

d) Show that the transfer function (matrix) for the closed loop system between input and output is given by

$$G(s) = \frac{Y(s)}{R(s)} = (C - Dk)(sI - (A - Bk))^{-1} BG_{pf} + DG_{pf}$$

and if  $D$  is zero this simplifies to

$$G(s) = \frac{Y(s)}{R(s)} = C(sI - (A - Bk))^{-1} BG_{pf}$$

e) Assume  $r(t) = u(t)$  and  $D = 0$ . Show that, in order for  $\lim_{t \rightarrow \infty} y(t) = 1$ , we must have

$$G_{pf} = \frac{-1}{C(A - Bk)^{-1}B}$$

Note that the prefilter gain is a function of the state variable feedback gain!

If matrix  $P$  is given as

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the determinant of  $P$  is given by  $ad - bc$ . This determinant will also give us the characteristic polynomial of the system.

2) For each of the systems below:

- determine the transfer function when there is state variable feedback
- determine if  $k_1$  and  $k_2$  exist ( $k = [k_1 \ k_2]$ ) to allow us to place the closed loop poles anywhere. That is, can we make the denominator look like  $s^2 + a_1s + a_0$  for any  $a_1$  and any  $a_0$ . If this is true, the system is said to be **controllable**.

a) Show that for

$$\begin{aligned} \dot{q} &= \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [0 \ 1]q + [0]u \end{aligned}$$

the closed loop transfer function with state variable feedback is  $G(s) = \frac{(s-1)G_{pf}}{(s-1)(s-1+k_2)}$

b) Show that for

$$\begin{aligned} \dot{q} &= \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= [0 \ 1]q + [0]u \end{aligned}$$

the closed loop transfer function with state variable feedback is  $G(s) = \frac{sG_{pf}}{s^2 + (k_2 - 1)s + k_1}$

c) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

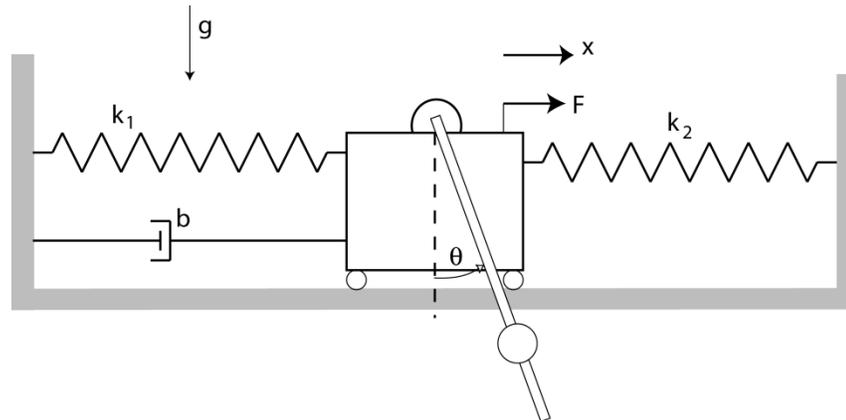
$$y = [1 \quad 0] q + [0] u$$

the closed loop transfer function with state variable feedback is  $G(s) = \frac{G_{pf}}{s^2 + (k_2 - 1)s + (k_1 - 1)}$

**The remaining problems are the Prelab for Lab 5 (No Maple, to be turned in with your homework)**

3) In this derivation we will make a state variable model for a regular pendulum (a pendulum hanging down) attached to the first cart. It will be easier to measure the parameters for a regular pendulum since it is a stable system. In the lab we will initially try and control the regular pendulum. Once this is working, we will try to control an inverted pendulum. To go from the model of a regular pendulum to the model of an inverted pendulum we use the substitution  $l \rightarrow -l$ .

Consider the configuration for the regular pendulum shown below:



The equations of motion for the regular pendulum can be written as

$$(J + ml^2)\ddot{\theta} + ml \cos(\theta)\ddot{x} + mgl \sin(\theta) = 0$$

$$(M + m)\ddot{x} + ml\ddot{\theta} \cos(\theta) - ml\dot{\theta}^2 \sin(\theta) + c\dot{x} + kx = F$$

The mass of the cart is  $M$ , the mass at the center of mass of the pendulum is  $m$ , the moment of inertial of the pendulum about its center of mass is  $J$ ,  $L$  is the length of the pendulum, and  $l$  is the distance from the pivot to the center of mass of the pendulum. The angle  $\theta$  is measured counterclockwise from straight up,  $x$  is the displacement of the first cart (positive to the right), and  $g$  is the gravitational constant.

a) Using a small angle/small velocity assumption, show that we can approximate the above equations of motion as

$$\begin{aligned}(J + ml^2)\ddot{\theta} + ml\ddot{x} + mgl\theta &\approx 0 \\ (M + m)\ddot{x} + ml\ddot{\theta} + c\dot{x} + kx &\approx F\end{aligned}$$

b) We can rewrite the first equation above as

$$\frac{1}{\omega_\theta^2}\ddot{\theta} + \frac{1}{g}\ddot{x} + \theta = 0$$

What is  $\omega_\theta^2$ ?

c) If we assume the cart is fixed, then  $\ddot{x} = 0$  and we have

$$\ddot{\theta} + \omega_\theta^2\theta = 0$$

This is the equation for a simple pendulum. If the pendulum is deflected a small angle and released, it will oscillate with frequency  $\omega_\theta$ . If we measure the period of the oscillations  $T_\theta$  how do we find  $\omega_\theta$ ?

d) We can rewrite the second equation from (a) as

$$\frac{1}{\omega_1^2}\ddot{x} + \frac{2\zeta_1}{\omega_1}\dot{x} + x + K_1\ddot{\theta} = K_2F$$

Find expressions for  $\omega_1$ ,  $\zeta_1$ ,  $K_1$ , and  $K_2$  in terms of  $m$ ,  $M$ ,  $k$ ,  $c$ , and  $l$ . If we assume there is no input ( $F = 0$ ) and the pendulum does not move very much ( $\ddot{\theta} \approx 0$ ) then we can use the log-decrement method to get initial estimates of  $\omega_1$  and  $\zeta_1$ .

e) Assuming we apply a step input of amplitude  $A$  to the cart, show that in steady state we get

$$K_2 = \frac{x_{ss}}{A}$$

f) Show that

$$\frac{\Theta(s)}{X(s)} = -\frac{\omega_\theta^2}{g} \left( \frac{s^2}{s^2 + \omega_\theta^2} \right)$$

We need to measure the gravitational constant in cm, since all other distances are measured in cm.

g) Show that

$$\frac{X(s)}{F(s)} = \frac{\omega_1^2 K_2 (s^2 + \omega_\theta^2)}{\left(1 - K_1 \omega_1^2 \frac{\omega_\theta^2}{g}\right) s^4 + (2\zeta_1 \omega_1) s^3 + (\omega_1^2 + \omega_\theta^2) s^2 + (2\zeta_1 \omega_1 \omega_\theta^2) s + \omega_1^2 \omega_\theta^2}$$

We can use this expression to determine  $K_1$  and get better estimates of  $\omega_1$  and  $\zeta_1$ .

**h)** We can rewrite our linearized dynamical equations as

$$\begin{aligned}\ddot{\theta} &\approx -\frac{1}{g}\omega_{\theta}^2\ddot{x} - \omega_{\theta}^2\theta \\ \ddot{x} &\approx -2\zeta_1\omega_1\dot{x} - \omega_1^2x - K_1\omega_1^2\ddot{\theta} + K_2\omega_1^2F\end{aligned}$$

By substituting the second equation into the first equation, show that we get

$$\ddot{\theta} \approx \frac{1}{\Delta} \left( \frac{\omega_1^2 \omega_{\theta}^2}{g} \right) x + \frac{1}{\Delta} \left( \frac{2\zeta_1 \omega_1 \omega_{\theta}^2}{g} \right) \dot{x} + \frac{1}{\Delta} (-\omega_{\theta}^2) \theta + \frac{1}{\Delta} \left( -\frac{\omega_1^2 \omega_{\theta}^2 K_2}{g} \right) F$$

where

$$\Delta = 1 - \left( \frac{K_1 \omega_1^2 \omega_{\theta}^2}{g} \right)$$

and substituting the first equation into the second equation we get

$$\ddot{x} \approx \frac{1}{\Delta} (-\omega_1^2) x + \frac{1}{\Delta} (-2\zeta_1 \omega_1) \dot{x} + \frac{1}{\Delta} (K_1 \omega_1^2 \omega_{\theta}^2) \theta + \frac{1}{\Delta} (K_2 \omega_1^2) F$$

**i)** Defining  $q_1 = x$ ,  $q_2 = \dot{x}$ ,  $q_3 = \theta$ , and  $q_4 = \dot{\theta}$ , show that we get the following state equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{\omega_1^2}{\Delta}\right) & -\left(\frac{2\zeta_1 \omega_1}{\Delta}\right) & \left(\frac{K_1 \omega_1^2 \omega_{\theta}^2}{\Delta}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\frac{\omega_1^2 \omega_{\theta}^2}{g\Delta}\right) & \left(\frac{2\zeta_1 \omega_1 \omega_{\theta}^2}{g\Delta}\right) & -\left(\frac{\omega_{\theta}^2}{\Delta}\right) & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{K_2 \omega_1^2}{\Delta}\right) \\ 0 \\ -\left(\frac{\omega_1^2 \omega_{\theta}^2 K_2}{g\Delta}\right) \end{bmatrix} F$$

**j)** If we now want to model the inverted pendulum ( $l \rightarrow -l$  to go back to the inverted pendulum), which terms change in the matrices above?

**k)** When we try and fit the frequency response data we see in the lab we will often get an unusual response. To understand this response we will analytically try and show what is happening. If you've not screwed up, you should have obtained values of

$$\begin{aligned}K_1 &= \frac{ml}{k} \\ \omega_1^2 &= \frac{k}{M+m} \\ \omega_{\theta}^2 &= \frac{mgl}{J+ml^2}\end{aligned}$$

If we assume that the mass of the cart and pendulum attachment is much larger than the mass at the center of

mass of the pendulum, then we have  $M$  is much larger than  $m$ . Secondly,  $J$  is the moment of inertia about the center of mass of the pendulum,  $m$  is the mass at the center of mass of the pendulum, and  $l$  is the distance from the pivot to the center of mass of the pendulum. For our systems, the pendulum bars have negligible mass and all of the mass is essentially concentrated at the center of mass. Hence  $ml^2$  is much larger than  $J$ . Using these two assumptions, show that  $\Delta \approx 1$ .

4) Assuming  $\Delta \approx 1$ , show that 
$$\frac{X(s)}{F(s)} \approx \frac{\omega_1^2 K_2}{s^2 + 2\zeta_1 \omega_1 s + \omega_1^2}$$

That is, there is a pole/zero cancellation! This is the effect you will often see in lab.

5) Modify *Basic\_2dof\_State\_Variable\_Model.mdl*, and *Basic\_2dof\_State\_Variable\_Model\_Driver.m* as necessary to work with the **regular** pendulum model. (These files and the state model for the regular pendulum are available on the course website.) Specifically, you need to

- Set the input of the system to zero (this is a regulator problem, in that we are just trying to hold the pendulum in place)
- Set the initial value of the pendulum to 0.05 radians and all other initial conditions to zero.
- Utilize pole placement to control the position of the pendulum and the cart. The goal is to keep the pendulum pointing straight down and keep the cart from moving more than about 2.5 cm in each direction. The control effort should also be less than 0.4 and the system should come to steady state in less than 1.0 seconds.

*You will need to turn in you plot, your Simulink code, and your Matlab code.*

6) Utilize the results of problem 5 to model the **inverted** pendulum. The only thing you should need to change is the state model. (The state model is available on the course website.) Specifically, you need to

- Set the input of the system to zero (this is a regulator problem, in that we are just trying to hold the pendulum in place)
- Set the initial value of the pendulum to 0.05 radians and all other initial conditions to zero.
- Utilize pole placement to control the position of the pendulum and the cart. The goal is to keep the pendulum pointing straight up and keep the cart from moving more than about 2.5 cm in each direction. The control effort should also be less than 0.4. Limiting the cart motion is usually the most difficult part. Often your controller for the regular pendulum will work for this part too, but not always.