

**ECE-320: Linear Control Systems**  
Homework 6

Due: Thursday April 19, 2012 at the beginning of class

1) For one of the rectilinear systems in lab, I found the following state variable representations:

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ -174.8205 & -2.6469 \end{bmatrix} q + \begin{bmatrix} 0 \\ 5050 \end{bmatrix} u$$
$$y = [1 \quad 0]q + [0]u$$

The closed loop transfer function with state variable feedback is

$$G_o(s) = \frac{5050G_{pf}}{s^2 + (2.6469 + 5050k_2)s + (174.8205 + 5050k_1)}$$

a) **Design I**

- Determine feedback gains  $k_1$  and  $k_2$  so the resulting system has real poles and a bandwidth of 2 Hz. Set the second pole twice as far from the origin as the first pole.
- Use the final value Theorem to determine the prefilter gain so the steady state error for a step input is zero.
- Estimate the settling time for the step response based on the system bandwidth.
- Use Matlab to plot the Bode plot of the closed loop system (to verify the bandwidth) and plot the step response (to verify the estimated settling time).

*You should get numbers like 0.028, 0.0069, 0.06, and 0.32. When you look at the Bode plot, the bandwidth will not be 2 Hz, primarily because the second pole is too close to the first pole.*

b) **Design II**

- Determine feedback gains  $k_1$  and  $k_2$  so the resulting (ideal second order) system has a percent overshoot of 15% and a settling time of 0.5 seconds.
- Use the final value Theorem to determine the prefilter gain so the steady state error for a step input is zero.
- Use Matlab to plot the step response (to verify the percent overshoot and settling time).
- Determine (analytically) the steady state output if the input to the closed loop system is  $r(t) = 1 \cos(4t)$  cm and determine the time delay between the input and the output.

*You should get numbers like 0.0128, 0.0026, 0.047, and a time delay of 0.07 seconds. Your step response should be fairly close to the requirements.*

2) (Easy) Show that  $\sum_{l=-\infty}^{l=n} \delta(l) = u(n)$  and  $\sum_{l=-\infty}^{l=n} \delta(l-k) = u(n-k)$

3) (Easy) For impulse response  $h(n) = \delta(n) + 2\delta(n-2) + 3\delta(n-3)$  and input

$x(n) = \left(\frac{1}{2}\right)^{n-1} u(n-2)$ , determine the output  $y(n)$  (this should be easy).

4) For impulse response  $h(n) = \left(\frac{1}{2}\right)^n u(n)$  and input  $x(n) = u(n)$ , show that the system output is

$$y(n) = 2 \left[ 1 - \left(\frac{1}{2}\right)^{n+1} \right] u(n)$$

a) by evaluating the convolution sum  $y(n) = \sum_{k=-\infty}^{k=\infty} x(n-k)h(k)$

b) by evaluating the convolution sum  $y(n) = \sum_{k=-\infty}^{k=\infty} x(k)h(n-k)$

*Note that this is the unit step response of the system.*

5) For impulse response  $h(n) = \left(\frac{1}{3}\right)^{n-2} u(n-1)$  and input  $x(n) = \left(\frac{1}{2}\right)^n u(n-1)$ , show that the system

output is  $y(n) = 9 \left[ \left(\frac{1}{2}\right)^{n-1} - \left(\frac{1}{3}\right)^{n-1} \right] u(n-2)$  by evaluating the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k)$$

6) For impulse response  $h(n) = \left(\frac{1}{2}\right)^{n-3} u(n-1)$  and input  $x(n) = \left(\frac{1}{4}\right)^{n+1} u(n-2)$ , show that the system

output is  $y(n) = \left[ \left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^{n-1} \right] u(n-3)$  by evaluating the convolution sum  $y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$

7) For impulse response  $h(n) = \left(\frac{1}{3}\right)^{n+1} u(n-2)$  and input  $x(n) = \left(\frac{1}{2}\right)^{n-2} u(n+1)$ , show that the system output is  $y(n) = \frac{16}{9} \left[ \left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n \right] u(n-1)$  by evaluating the convolution sum  $y(n) = \sum_{k=-\infty}^{\infty} h(n-k)x(k)$

8) For  $h(n) = \left(\frac{1}{a}\right)^n u(-n) + a^n u(n)$  with  $|a| < 1$ , using the two-sided  $z$  transform to show that

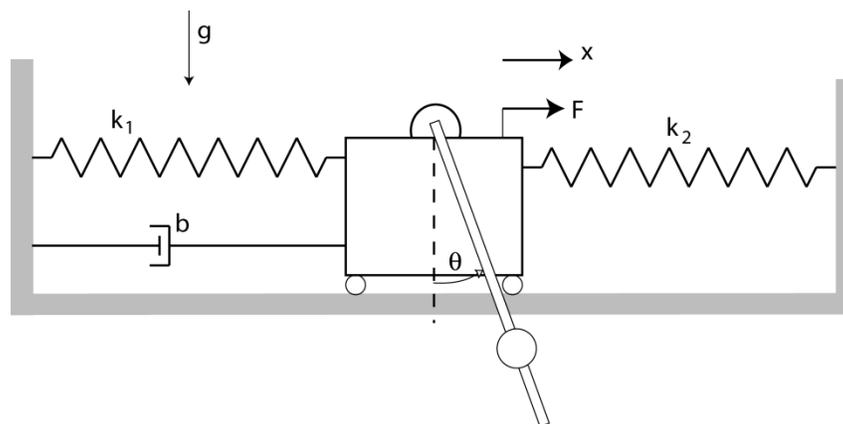
$$H(z) = \frac{1}{1-az} + \frac{1}{1-az^{-1}}$$

and the region of convergence is  $|a| < |z| < \frac{1}{|a|}$ .

**The remaining problems are the Prelab for Lab 5 (No Maple)**

9) In this derivation we will make a state variable model for a regular pendulum (a pendulum hanging down) attached to the first cart. It will be easier to measure the parameters for a regular pendulum since it is a stable system. In the lab we will initially try and control the regular pendulum. Once this is working, we will try to control an inverted pendulum. To go from the model of a regular pendulum to the model of an inverted pendulum we use the substitution  $l \rightarrow -l$ .

Consider the configuration for the regular pendulum shown below:



The equations of motion for the regular pendulum can be written as

$$\begin{aligned}(J + ml^2)\ddot{\theta} + ml \cos(\theta)\ddot{x} + mgl \sin(\theta) &= 0 \\ (M + m)\ddot{x} + ml\ddot{\theta} \cos(\theta) - ml\dot{\theta}^2 \sin(\theta) + c\dot{x} + kx &= F\end{aligned}$$

The mass of the cart is  $M$ , the mass at the center of mass of the pendulum is  $m$ , the moment of inertia of the pendulum about its center of mass is  $J$ ,  $L$  is the length of the pendulum, and  $l$  is the distance from the pivot to the center of mass of the pendulum. The angle  $\theta$  is measured counterclockwise from straight up,  $x$  is the displacement of the first cart (positive to the right), and  $g$  is the gravitational constant.

a) Using a small angle/small velocity assumption, show that we can approximate the above equations of motion as

$$\begin{aligned}(J + ml^2)\ddot{\theta} + ml\ddot{x} + mgl\theta &\approx 0 \\ (M + m)\ddot{x} + ml\ddot{\theta} + c\dot{x} + kx &\approx F\end{aligned}$$

b) We can rewrite the first equation above as

$$\frac{1}{\omega_\theta^2}\ddot{\theta} + \frac{1}{g}\ddot{x} + \theta = 0$$

What is  $\omega_\theta^2$ ?

c) If we assume the cart is fixed, then  $\ddot{x} = 0$  and we have

$$\ddot{\theta} + \omega_\theta^2\theta = 0$$

This is the equation for a simple pendulum. If the pendulum is deflected a small angle and released, it will oscillate with frequency  $\omega_\theta$ . If we measure the period of the oscillations  $T_\theta$  how do we find  $\omega_\theta$ ?

d) We can rewrite the second equation from (a) as

$$\frac{1}{\omega_1^2}\ddot{x} + \frac{2\zeta_1}{\omega_1}\dot{x} + x + K_1\ddot{\theta} = K_2F$$

Find expressions for  $\omega_1$ ,  $\zeta_1$ ,  $K_1$ , and  $K_2$  in terms of  $m$ ,  $M$ ,  $k$ ,  $c$ , and  $l$ . If we assume there is no input ( $F = 0$ ) and the pendulum does not move very much ( $\ddot{\theta} \approx 0$ ) then we can use the log-decrement method to get initial estimates of  $\omega_1$  and  $\zeta_1$ .

e) Assuming we apply a step input of amplitude  $A$  to the cart, show that in steady state we get

$$K_2 = \frac{x_{ss}}{A}$$

f) Show that

$$\frac{\Theta(s)}{X(s)} = -\frac{\omega_\theta^2}{g} \left( \frac{s^2}{s^2 + \omega_\theta^2} \right)$$

We need to measure the gravitational constant in cm, since all other distances are measured in cm.

g) Show that

$$\frac{X(s)}{F(s)} = \frac{\omega_1^2 K_2 (s^2 + \omega_\theta^2)}{\left(1 - K_1 \omega_1^2 \frac{\omega_\theta^2}{g}\right) s^4 + (2\zeta_1 \omega_1) s^3 + (\omega_1^2 + \omega_\theta^2) s^2 + (2\zeta_1 \omega_1 \omega_\theta^2) s + \omega_1^2 \omega_\theta^2}$$

We can use this expression to determine  $K_1$  and get better estimates of  $\omega_1$  and  $\zeta_1$ .

h) We can rewrite our linearized dynamical equations as

$$\begin{aligned}\ddot{\theta} &\approx -\frac{1}{g} \omega_\theta^2 \ddot{x} - \omega_\theta^2 \theta \\ \ddot{x} &\approx -2\zeta_1 \omega_1 \dot{x} - \omega_1^2 x - K_1 \omega_1^2 \ddot{\theta} + K_2 \omega_1^2 F\end{aligned}$$

By substituting the second equation into the first equation, show that we get

$$\ddot{\theta} \approx \frac{1}{\Delta} \left( \frac{\omega_1^2 \omega_\theta^2}{g} \right) x + \frac{1}{\Delta} \left( \frac{2\zeta_1 \omega_1 \omega_\theta^2}{g} \right) \dot{x} + \frac{1}{\Delta} (-\omega_\theta^2) \theta + \frac{1}{\Delta} \left( -\frac{\omega_1^2 \omega_\theta^2 K_2}{g} \right) F$$

where

$$\Delta = 1 - \left( \frac{K_1 \omega_1^2 \omega_\theta^2}{g} \right)$$

and substituting the first equation into the second equation we get

$$\ddot{x} \approx \frac{1}{\Delta} (-\omega_1^2) x + \frac{1}{\Delta} (-2\zeta_1 \omega_1) \dot{x} + \frac{1}{\Delta} (K_1 \omega_1^2 \omega_\theta^2) \theta + \frac{1}{\Delta} (K_2 \omega_1^2) F$$

i) Defining  $q_1 = x$ ,  $q_2 = \dot{x}$ ,  $q_3 = \theta$ , and  $q_4 = \dot{\theta}$ , show that we get the following state equations

$$\begin{bmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{q}_3 \\ \dot{q}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\left(\frac{\omega_1^2}{\Delta}\right) & -\left(\frac{2\zeta_1 \omega_1}{\Delta}\right) & \left(\frac{K_1 \omega_1^2 \omega_\theta^2}{\Delta}\right) & 0 \\ 0 & 0 & 0 & 1 \\ \left(\frac{\omega_1^2 \omega_\theta^2}{g\Delta}\right) & \left(\frac{2\zeta_1 \omega_1 \omega_\theta^2}{g\Delta}\right) & -\left(\frac{\omega_\theta^2}{\Delta}\right) & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{bmatrix} + \begin{bmatrix} 0 \\ \left(\frac{K_2 \omega_1^2}{\Delta}\right) \\ 0 \\ -\left(\frac{\omega_1^2 \omega_\theta^2 K_2}{g\Delta}\right) \end{bmatrix} F$$

j) If we now want to model the inverted pendulum ( $l \rightarrow -l$  to go back to the inverted pendulum), which terms change in the matrices above?

k) When we try and fit the frequency response data we see in the lab we will often get an unusual response. To understand this response we will analytically try and show what is happening. If you've not screwed up, you should have obtained values of

$$K_1 = \frac{ml}{k}$$

$$\omega_1^2 = \frac{k}{M+m}$$

$$\omega_\theta^2 = \frac{mgl}{J+ml^2}$$

If we assume that the mass of the cart and pendulum attachment is much larger than the mass at the center of Mass of the pendulum, then we have  $M$  is much larger than  $m$ . Secondly,  $J$  is the moment of inertia about the center of mass of the pendulum,  $m$  is the mass at the center of mass of the pendulum, and  $l$  is the distance from the pivot to the center of mass of the pendulum. For our systems, the pendulum bars have negligible mass and all of the mass is essentially concentrated at the center of mass. Hence  $ml^2$  is much larger than  $J$ . Using these two assumptions, show that  $\Delta \approx 1$ .

10) Assuming  $\Delta \approx 1$ , show that  $\frac{X(s)}{F(s)} \approx \frac{\omega_1^2 K_2}{s^2 + 2\zeta_1 \omega_1 s + \omega_1^2}$

That is, there is a pole/zero cancellation! This is the effect you will often see in lab.

11) Modify *Basic\_2dof\_State\_Variable\_Model.mdl*, and *Basic\_2dof\_State\_Variable\_Model\_Driver.m* as necessary to work with the **regular** pendulum model. (These files and the state model for the regular pendulum are available on the course website.) Specifically, you need to

- Set the input of the system to zero (this is a *regulator problem*, in that we are just trying to hold the pendulum in place)
- Set the initial value of the pendulum to 0.05 radians and all other initial conditions to zero.
- Utilize pole placement to control the position of the pendulum and the cart. The goal is to keep the pendulum pointing straight down and keep the cart from moving more than about 2.5 cm in each direction. The control effort should also be less than 0.4 and the system should come to steady state in less than 1.0 seconds.

*You will need to turn in you plot, your Simulink code, and your Matlab code.*

12) Utilize the results of problem 11 to model the **inverted** pendulum. The only thing you should need to change is the state model. (The state model is available on the course website.) Specifically, you need to

- Set the input of the system to zero (this is a *regulator problem*, in that we are just trying to hold the pendulum in place)
- Set the initial value of the pendulum to 0.05 radians and all other initial conditions to zero.

- Utilize pole placement to control the position of the pendulum and the cart. The goal is to keep the pendulum pointing straight up and keep the cart from moving more than about 2.5 cm in each direction. The control effort should also be less than 0.4. Limiting the cart motion is usually the most difficult part. Often your controller for the regular pendulum will work for this part too, but not always.