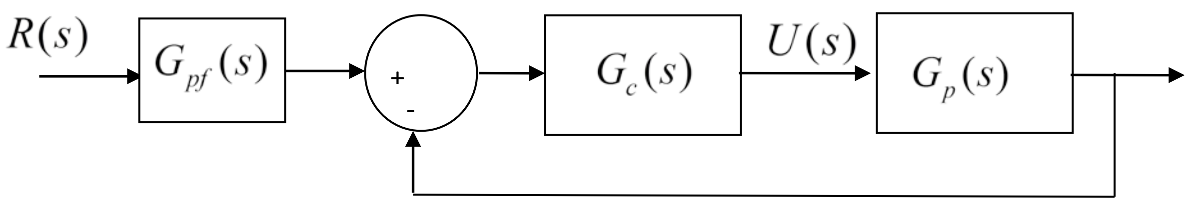
ECE-320: Linear Control Systems
Homework 5

Due: Thursday April 11 at the beginning of class

1) One of the things that will be coming up in lab more and more is the limitation of the amplitude of the control signal, or the *control effort*. This is also a problem for most practical systems. In this problem we will do some simple analysis to better understand why Matlab's sisotool won't give us a good estimate of the control effort for some types of systems, and why dynamic prefilters can often really help us out here.

a) For the system below,



show that U(s) and R(s) are related by

$$U(s) = \frac{G_c(s)G_{pf}(s)}{1+G_p(s)G_c(s)}R(s)$$

b) For many types of controllers, the maximum value of the control signal is just after the step is applied, at $t = 0^+$. Although most of the time we are concerned with steady state values and use the final value Theorem in the *s*-plane, in this case we want to use the initial value Theorem, which can be written as

$$\lim_{t\to 0^+} u(t) = \lim_{s\to\infty} sU(s)$$

If the system input is a step of amplitude *A*, show that

$$u(0^+) = \lim_{s\to\infty} \frac{AG_c(s)G_{pf}(s)}{1+G_p(s)G_c(s)}$$

This result shows very clearly that the initial control signal is directly proportional to the amplitude of the input signal, which is pretty intuitive.

c) Now let's assume

$$G_c(s) = \frac{N_c(s)}{D_c(s)} \quad G_p(s) = \frac{N_p(s)}{D_p(s)} \quad G_{pf}(s) = \frac{N_{pf}(s)}{D_{pf}(s)}$$

If we want to look at the initial value for a unit step, we need to look at

$$u(0^+) = \lim_{s\to\infty} \frac{sG_c(s)G_{pf}(s)}{1+G_c(s)G_p(s)}\frac{1}{s} = \lim_{s\to\infty} \frac{G_c(s)G_{pf}(s)}{1+G_c(s)G_p(s)}$$

Let's also then define

$$\tilde{U}(s) = \frac{G_c(s)G_{pf}(s)}{1+G_c(s)G_p(s)}$$

so that

$$u(0^+) = \lim_{s\to\infty} \tilde{U}(s)$$

Show that

$$\tilde{U}(s) = \frac{N_{pf}(s)}{\left(\frac{D_c(s)}{N_c(s)}\right)D_{pf}(s) + \left(\frac{N_p(s)}{D_p(s)}\right)D_{pf}(s)}$$

and

$$\deg \tilde{U} = \deg N_{pf} - \max\left[\deg D_c - \deg N_c + \deg D_{pf}, \deg N_p - \deg D_p + \deg D_{pf}\right]$$

where $\deg Y$ is the degree of polynomial Y.

d) Since we are going to take the limit as $s \to \infty$, we need the degree of $\tilde{U}(s)$ to be less than or equal to zero for a step input to have a finite $u(0^+)$. Why?

For our 1 dof systems in lab, we have $\deg N_p = 0$ *and* $\deg D_p = 2$. *Use this for the remainder of this problem*

e) If the prefilter is a constant, show that in order to have a finite $u(0^+)$ we must have

$$\deg D_c \geq \deg N_c$$

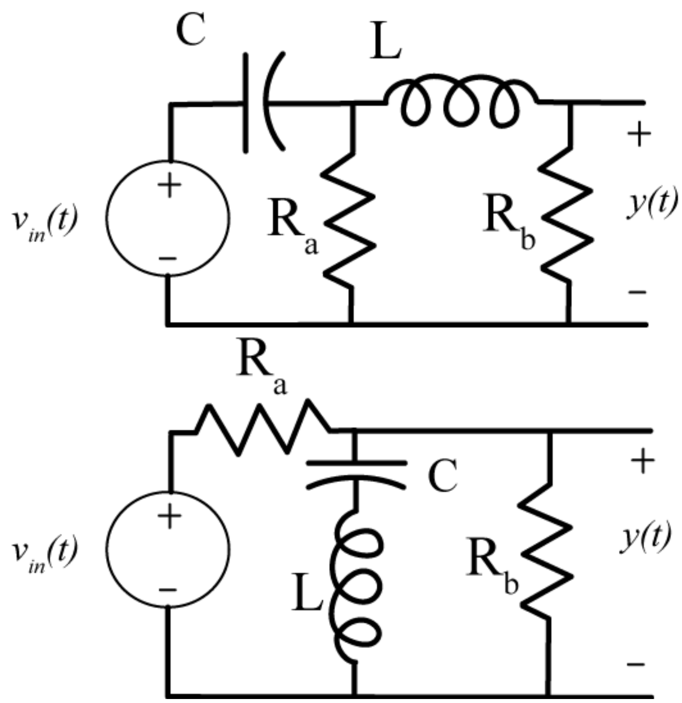
f) If the numerator of the prefilter is a constant, then in order to have a finite $u(0^+)$ we must have

$$\deg D_c - \deg N_c + \deg D_{pf} \geq 0 \text{ or } -2 + \deg D_{pf} \geq 0$$

g) For P, I, D, PI, PD, PID, and lead controllers, determine if $u(0^+)$ is finite if the prefilter is a constant.

Note: Although it may appear that the control effort is sometimes infinite, in practice this is not true since our motor cannot produce an infinite signal. This large initial control signal is referred to as a *set-point kick*. There are different ways to implement a PID controller to avoid this, and we will cover two of them in Lab 4.

2) For the following two circuits,



show that the state variable descriptions are given by

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_b}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_a C} \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ \frac{1}{R_a C} \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} R_B & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + [0] v_{in}(t)$$

$$\frac{d}{dt}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_a R_b}{L(R_a + R_b)} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_b}{L(R_a + R_b)} \\ 0 \end{bmatrix} v_{in}(t) \quad y(t) = \begin{bmatrix} -\frac{R_a R_b}{R_a + R_b} & 0 \end{bmatrix}\begin{bmatrix} i_L(t) \\ v_c(t) \end{bmatrix} + \begin{bmatrix} \frac{R_b}{R_a + R_b} \end{bmatrix} v_{in}(t)$$

3) For the plant

$$G_p(s) = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

a) If the plant input is $u(t)$ and the output is $x(t)$, show that we can represent this system with the differential equation

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = K\omega_n^2 u(t)$$

b) Assuming we use states $q_1(t) = x(t)$ and $q_2(t) = \dot{x}(t)$, and the output is $x(t)$, show that we can write the state variable description of the system as

$$\frac{d}{dt}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ K\omega_n^2 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} q_1(t) \\ q_2(t) \end{bmatrix} + \begin{bmatrix} 0 \end{bmatrix} u(t)$$

or

$$\dot{q}(t) = Aq(t) + Bu(t) \quad y(t) = Cq(t) + Du(t)$$

Determine the A, B, C and D matrices.

c) Assume we use state variable feedback of the form $u(t) = G_{pf}r(t) - kq(t)$, where $r(t)$ is the new input to the system, G_{pf} is a prefilter (for controlling the steady state error), and k is the state variable feedback gain vector. Show that the state variable model for the closed loop system is

$$\dot{q}(t) = (A - Bk)q(t) + (BG_{pf})r(t)$$
$$y(t) = (C - Dk)q(t) + (DG_{pf})r(t)$$

or

$$\dot{q}(t) = \tilde{A}q(t) + \tilde{B}r(t)$$
$$y(t) = \tilde{C}q(t) + \tilde{D}r(t)$$

d) Show that the transfer function (matrix) for the closed loop system between input and output is given by

$$G(s) = \frac{Y(s)}{R(s)} = (C - Dk)(sI - (A - Bk))^{-1}BG_{pf} + DG_{pf}$$

and if D is zero this simplifies to

$$G(s) = \frac{Y(s)}{R(s)} = C(sI - (A - Bk))^{-1}BG_{pf}$$

e) Assume $r(t) = u(t)$ and $D = 0$. Show that, in order for $\lim_{t\to\infty} y(t) = 1$, we must have

$$G_{pf} = \frac{-1}{C(A - Bk)^{-1}B}$$

Note that the prefilter gain is a function of the state variable feedback gain!

If matrix P is given as

$$P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then

$$P^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and the determinant of P is given by $ad - bc$. This determinant will also give us the characteristic polynomial of the system.

4) For each of the systems below:

- determine the transfer function when there is state variable feedback
- determine if k_1 and k_2 exist ($k = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$) to allow us to place the closed loop poles anywhere. That is, can we make the denominator look like $s^2 + a_1 s + a_0$ for any a_1 and any a_0. If this is true, the system is said to be ***controllable***.

a) Show that for

$$\dot{q} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{(s-1)G_{pf}}{(s-1)(s-1+k_2)}$

b) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 0 & 1 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{sG_{pf}}{s^2 + (k_2 - 1)s + k_1}$

c) Show that for

$$\dot{q} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} q + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$
$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} q + [0]u$$

the closed loop transfer function with state variable feedback is $G(s) = \dfrac{G_{pf}}{s^2 + (k_2 - 1)s + (k_1 - 1)}$