ECE-320

Equation Sheet

Second Order System Properties

Percent Overshoot: $P.O. = e^{\frac{-\zeta\pi}{\sqrt{1-\zeta^2}}} \times 100\%$, If $\beta = \frac{PO^{max}}{100}$ then $\zeta = \frac{\frac{-\ln(\beta)}{\pi}}{\sqrt{1+\left(\frac{-\ln(\beta)}{\pi}\right)^2}}$, $\theta = cos^{-1}(\zeta)$

Time to Peak: $T_p = \frac{\pi}{\omega_d}$, $\omega_d = \omega_n\sqrt{1-\zeta^2}$ 2% Settling Time: $T_s = \frac{4}{\zeta\omega_n} = 4\tau$

Model Matching

Assume we have a proper plant $G_p(s) = N(s)/D(s)$ and we want the closed loop system to have the transfer function $G_o(s) = N_0(s)/D_0(s)$. We can find a controller

$$G_c(s) = \frac{N_0(s)D(s)}{N(s)\left[D_0(s) - N_0(s)\right]}$$

under the following conditions:

- degree $D_0(s)$ - degree $N_0(s) \geq$ degree $D(s)$ - degree $N(s)$
- The right half plane zeros of $N(s)$ are retained in $N_0(s)$
- $G_0(s)$ is stable

Controller Types

Proportional (P), $G_c(s) = k$

Integral (I), $G_c(s) = \frac{k}{s}$

Proportional + Integral (PI), $G_c(s) = \frac{k(s+z)}{s}$

Proportional + Derivative (PD), $G_c(s) = k(s+z)$

Proportional + Integral + Derivative (PID), $G_c(s) = \frac{k(s+z_1)(s+z_2)}{s}$

Lead, $G_c(s) = \frac{k(s+z)}{(s+p)}$, $p > z$

Lag, $G_c(s) = \frac{k(s+z)}{(s+p)}$, $z > p$

Root Locus Construction

Once each pole has been paired with a zero, we are done

1. *Loci Branches*

$$\underset{k=0}{poles} \rightarrow \underset{k=\infty}{zeros}$$

Continuous curves, which comprise the locus, start at each of the *n* poles of $G(s)$ for which $k=0$. As *k* approaches ∞ , the branches of the locus approach the *m* zeros of $G(s)$. Locus branches for excess poles extend to infinity.

The root locus is s<u>**ymmetric**</u> about the real axis.

2. *Real Axis Segments*

The root locus includes all points along the real axis to the left of an odd number of poles plus zeros of $G(s)$.

3. *Asymptotic Angles*

As $k \rightarrow \infty$ $k , the branches of the locus become asymptotic to straight lines with angles

$$\theta = \frac{180^o + i360^o}{n-m}, \; i = 0, \pm 1, \pm 2, ..$$

until all $(n-m)$ angles not differing by multiples of 360^o are obtained. *n* is the number of poles of $G(s)$ and *m* is the number of zeros of $G(s)$.

4. *Centroid of the Asymptotes*

The starting point on the real axis from which the asymptotic lines radiate is given by

$$\sigma_c = \frac{\sum_i p_i - \sum_j z_j}{n-m}$$

where p_i is the i^{th} pole of $G(s)$, z_j is the j^{th} zero of $G(s)$, *n* is the number of poles of $G(s)$ and *m* is the number of zeros of $G(s)$. This point is termed the *centroid of the asymptotes*.

5. *Leaving/Entering the Real Axis*

When two branches of the root locus leave or enter the real axis, they usually do so at angles of $\pm 90^o$.

Laplace Transforms

$$\mathcal{L}\{\delta(t)\} = 1$$

$$\mathcal{L}\{u(t)\} = \frac{1}{s}$$

$$\mathcal{L}\{tu(t)\} = \frac{1}{s^2}$$

$$\mathcal{L}\left\{\frac{t^{m-1}}{(m-1)!}u(t)\right\} = \frac{1}{s^m}$$

$$\mathcal{L}\{e^{-at}u(t)\} = \frac{1}{s+a}$$

$$\mathcal{L}\{te^{-at}u(t)\} = \frac{1}{(s+a)^2}$$

$$\mathcal{L}\left\{\frac{t^{(m-1)}}{(m-1)!}e^{-at}u(t)\right\} = \frac{1}{(s+a)^m}$$

$$\mathcal{L}\{\cos(\omega_0 t)u(t)\} = \frac{s}{s^2+\omega_0^2}$$

$$\mathcal{L}\{\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{s^2+\omega_0^2}$$

$$\mathcal{L}\{e^{-\alpha t}\cos(\omega_0 t)u(t)\} = \frac{s+\alpha}{(s+\alpha)^2+\omega_0^2}$$

$$\mathcal{L}\{e^{-\alpha t}\sin(\omega_0 t)u(t)\} = \frac{\omega_0}{(s+\alpha)^2+\omega_0^2}$$

$$\mathcal{L}\left\{\frac{dx(t)}{dt}\right\} = sX(s) - x(0^-)$$

$$\mathcal{L}\left\{\frac{d^2x(t)}{dt^2}\right\} = s^2X(s) - sx(0^-) - \dot{x}(0^-)$$

$$\mathcal{L}\{x(t-a)\} = e^{-as}X(s)$$

$$\mathcal{L}\{e^{-at}x(t)\} = X(s+a)$$

$$\mathcal{L}\left\{x\left(\frac{t}{a}\right), a > 0\right\} = aX(as)$$