ECE-320: Linear Control Systems
Homework 1

Due: Wednesday March 15 at 3:30 PM

1) The (one sided) Laplace transform is defined as

$$X(s) = \int_0^\infty x(t)e^{-st}dt$$

or the transform pair

$$x(t) \Leftrightarrow X(s)$$

a) By differentiating both sides of the above equation with respect to s, show that

$$-\frac{dX(s)}{ds} = \int_0^\infty tx(t)e^{-st}dt$$

$$\frac{d^2X(s)}{ds^2} = \int_0^\infty t^2x(t)e^{-st}dt$$

or

$$tx(t) \Leftrightarrow -\frac{dX(s)}{ds}$$

$$t^2x(t) \Leftrightarrow \frac{d^2X(s)}{ds^2}$$

b) By evaluating the integral, show that, if the real part of a is positive,

$$x(t) = e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$$

c) Combining parts (a) and (b), and a little bit of Maple free calculus,
show that

$$x(t) = e^{-at}u(t) \Leftrightarrow \frac{1}{s+a}$$

$$tx(t) = te^{-at}u(t) \Leftrightarrow \frac{1}{(s+a)^2}$$

$$\frac{1}{2}t^2x(t) = \frac{1}{2}t^2e^{-at}u(t) \Leftrightarrow \frac{1}{(s+a)^3}$$

The moral of this problem is if a pole is repeated in the Laplace (s) domain, we just multiply by t in the time domain to get the shape of the time response. (There is still some scaling involved, but we are mostly concerned with the shape of the signal.)

2) Starting from the definition of the Laplace transform, show

$$x(t-t_0) \Leftrightarrow e^{-st_0} X(s)$$

Note that in this problem, we are assuming $x(t) = x(t)u(t)$ and that $x(t-t_0) = x(t-t_0)u(t-t_0)$. That is, we assume $x(t)$ is zero for $t < 0$ and $x(t-t_0)$ is zero for $t < t_0$.

The moral of this problem is that any time you see an e^{-st_0} *in the Laplace domain, there is a delay, or transport lag, in the time domain.*

3) Starting from the definition of the Laplace transform, show $x(t)e^{-at} \Leftrightarrow X(s+a)$
Using this result, and completing the square in the denominator, show that

$$\frac{As+B}{s^2+2as+c} \Leftrightarrow e^{-at}\left[A\cos(bt) + \frac{B-Aa}{b}\sin(bt)\right]u(t)$$

where $b^2 = c - a^2$ and b^2 is assumed to be positive (complex conjugate roots)

4) Determine the impulse response of the following using partial fractions as necessary. You are expected to be able to do all of these with the Laplace transform Table in the notes, and the properties above. You may *check* your answers with Maple.

a) $H(s) = \dfrac{e^{-s}}{s^2+3}$ *(Hint: Ignore the exponential term (assume it is a 1) and find the inverse Laplace transform, then include the transport delay.)*

b) $H(s) = \dfrac{s+1}{s^2+2}$

c) $H(s) = \dfrac{s+1}{(s+2)^2+4}$

d) $H(s) = \dfrac{s+3}{(s+1)(s+2)}$

e) $H(s) = \dfrac{s}{(s+1)^2(s+3)}$

f) $H(s) = \dfrac{1}{(s+1)(s+2)(s+3)(s+4)}$

g) $H(s) = \dfrac{s}{(s^2+2)(s+1)}$

h) $H(s) = \dfrac{1}{s^2+s+1}$

i) $H(s) = \dfrac{s}{2s^2+s+3}$

j) $H(s) = \dfrac{s}{s^2 + 3s + 2}$

5) For systems with the following transfer functions:

$$H_a(s) = \frac{1}{s+2}$$

$$H_b(s) = \frac{s+6}{(s+2)(s+3)}$$

a) Determine the unit step and unit ramp response for each system using Laplace transforms. Your answer should be time domain functions $y_a(t)$ and $y_b(t)$.

b) From these time domain functions, determine the steady state errors for a unit step and unit ramp input.

c) Using the equations derived in class (and in the notes), determine the steady state errors for a unit step and a unit ramp input directly from the transfer functions.

The following Matlab code can be used to estimate the step and ramp response for 5 seconds for transfer function $H_b(s)$.

```
H = tf([1 6],[1 5 6]);                    % enter the transfer function
t = [0:0.01:5];                           % t goes from 0 to 5 by increments of 0.01
ustep = ones(1,length(t));                % the step input is all ones, u(t) = 1;
uramp = t;                                % the ramp input is has the input u(t) = t;
ystep = lsim(H,ustep,t);                  % find the step response
yramp = lsim(H,uramp,t);                  % find the ramp response
figure;                                   % make a new figure
orient tall                               % or orient landscape, use more of the page
subplot(2,1,1);                           % put two graphs on one piece of paper
plot(t,ustep,'.-',t,ystep,'-');           % plot input/output with different line types
grid;                                     % put a grid on the graph
legend('Step Input','Step Response',4);   % put a legend on the graph
subplot(2,1,2);                           % second of two graphs on one piece of paper
plot(t,uramp,'.-',t,yramp,'-');           % plot input/output with different line types
grid;                                     % put a grid on the graph
legend('Ramp Input','Ramp Response',4);   % put a legend on the graph
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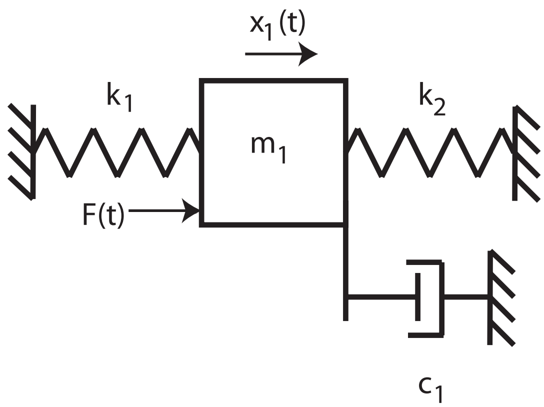
d) Plot the step and ramp response for both systems (a and b) and indicate the steady state errors on the graph. Draw on the graph to show you know what the steady state errors are.

Ans. Steady state errors for a unit step input: 0.5,0; for a unit ramp input : infinity and 0.666

Preparation for Lab 2

6) Consider the following one degree of freedom system we will be utilizing this term:



a) Draw a freebody diagram of the forces on the mass.

b) Show that the equations of motion can be written:

$$m_1\ddot{x}_1(t) + c_1\dot{x}(t) + (k_1 + k_2)x_1(t) = F(t)$$

or

$$\frac{1}{\omega_n^2}\ddot{x}_1(t) + \frac{2\zeta}{\omega_n}\dot{x}(t) + x_1(t) = KF(t)$$

c) What are the damping ratio ζ, the natural frequency ω_n, and the static gain K in terms of m_1, k_1, k_2, and c_1?

d) Show that the transfer function for the *plant* is given by

$$G_p(s) = \frac{X_1(s)}{F(s)} = \frac{K}{\frac{1}{\omega_n^2}s^2 + \frac{2\zeta}{\omega_n}s + 1}$$

7) One of the methods we will be using to identify ζ and ω_n is the *log-decrement* method, which we will review/derive in this problem. If our system is at rest and we provide the mass with an initial displacement away from equilibrium, the response due to this displacement can be written

$$x_1(t) = Ae^{-\zeta\omega_n t}\cos(\omega_d t + \theta)$$

where

$x_1(t)$ = displacement of the mass as a function of time
ζ = damping ratio
ω_n = natural frequency
ω_d = damped frequency = $\omega_n\sqrt{1-\zeta^2}$

After the mass is released, the mass will oscillate back and forth with period given by $T_d = \dfrac{2\pi}{\omega_d}$, so if we measure the period of the oscillation (T_d) we can estimate ω_d.

Let's assume t_0 is the time of one peak of the cosine. Since the cosine is periodic, subsequent peaks will occur at times given by $t_n = t_0 + nT_d$, where n is an integer.

a) Show that

$$\frac{x_1(t_0)}{x_1(t_n)} = e^{\zeta\omega_n T_d n}$$

b) If we define the log decrement as

$$\delta = \ln\left[\frac{x_1(t_0)}{x_1(t_n)}\right]$$

show that we can compute the damping ratio as

$$\zeta = \frac{\delta}{\sqrt{4n^2\pi^2 + \delta^2}}$$

c) Given the initial condition response shown in the Figures 3 and 4 on the next page, estimate the damping ratio and natural frequency using the log-decrement method. (*You should get answers that include the numbers 15, 0.2, 0.1 and 15, approximately.*)

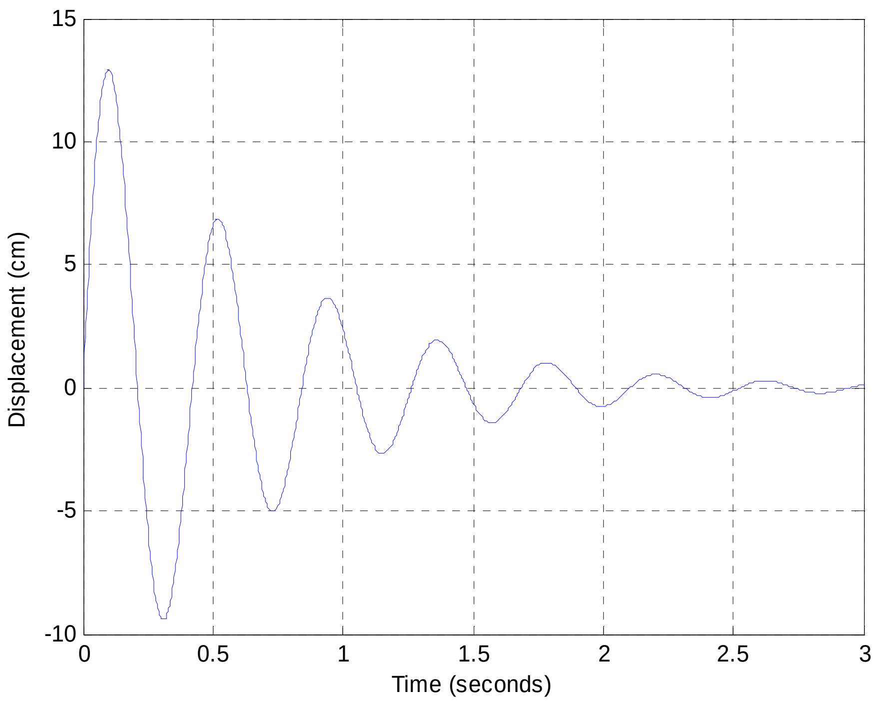


Figure 3. Initial condition response for second order system A.

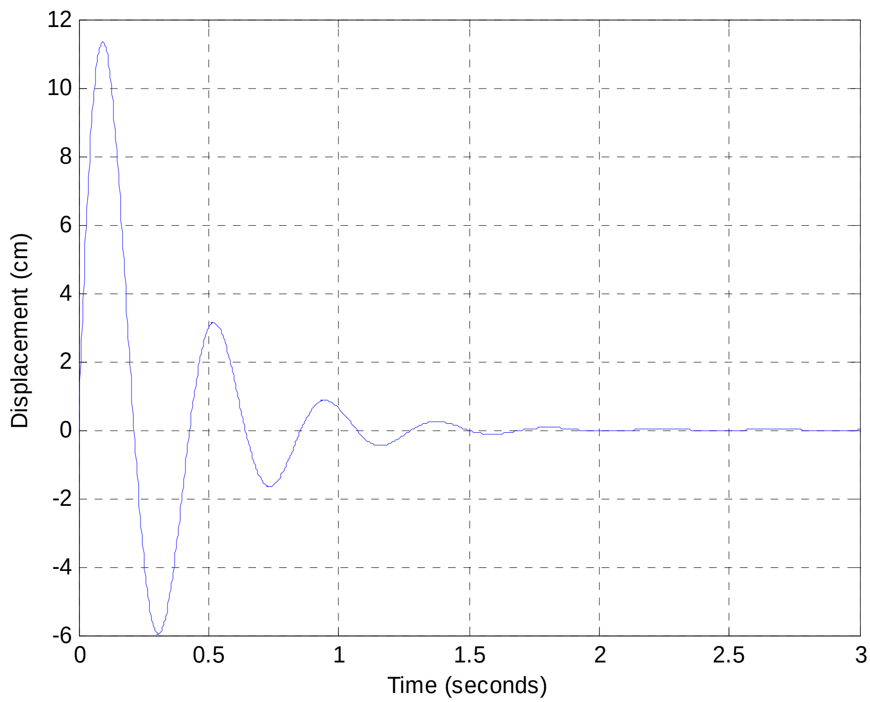


Figure 4. Initial condition response for second order system B.