ECE300 Hw 3 Solutions W0809

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10:35 AM

1) Determine the impulse response for following systems

a) $y(t) = \frac{1}{2}\left[x(t-1) + x(t+1)\right]$

$$\boxed{h(t) = \frac{1}{2}\left[\delta(t-1) + \delta(t+1)\right]}$$

(b) $y(t) = \int_{-\infty}^{t+1} e^{-(t-\lambda)} x(\lambda+3)\, d\lambda$

$$h(t) = \int_{-\infty}^{t+1} e^{-(t-\lambda)} \delta(\lambda+3)\, d\lambda$$

$$= e^{-(t+3)} \int_{-\infty}^{t+1} \delta(\lambda+3)\, d\lambda \longrightarrow \int \delta(\lambda+3)\, d\lambda = \begin{cases} 0 & t+1 < -3 \\ 1 & t+1 \geq -3 \end{cases} = u(t+4)$$

$$\boxed{= e^{-(t+3)} u(t+4)}$$

(c) $\dot{y}(t) + 2y(t) = 3x(t-1) \longrightarrow$ from class we solved for the $h(t)$ of this type of differential eqtn. However, for this case $x(t)$ is delayed, which will create a corresponding delay in $h(t)$.

for the non-delayed system w/ $x(t)$

$$h_1(t) = 3e^{-2t} u(t)$$

for the given system w/ $x(t-1)$

$$\boxed{h(t) = h_1(t-1) = 3e^{-2(t-1)} u(t-1)}$$

(d) $y(t) = x(t) + \int_{-\infty}^{t} e^{-2(t-\lambda)} x(\lambda)\, d\lambda$

$$h(t) = \delta(t) + \int_{-\infty}^{t} e^{-2(t-\lambda)} \delta(\lambda)\, d\lambda$$

$$= \delta(t) + e^{-2t} \int_{-\infty}^{t} \delta(\lambda)\, d\lambda = \boxed{\delta(t) + e^{-2t} u(t)}$$

Problem 2

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Given: $y(t) = \frac{1}{I}\int_{t-I}^{t} x(\lambda)\,d\lambda$

a) find Impulse response

$$h(t) = \frac{1}{I}\int_{t-I}^{t} \delta(\lambda)\,d\lambda = \begin{cases} 0 & t<0 \ \& \ t>I \\ \frac{1}{I} & 0 \le t \le I \end{cases}$$

$$= \frac{1}{I}\left[u(t) - u(t-I)\right]$$

(b) Find the STEP Response

$$s(t) = \frac{1}{I}\int_{t-I}^{t} u(\lambda)\,d\lambda = \begin{cases} 0 & t<0 \\ \frac{1}{I}\int_{0}^{t} d\lambda & 0<t<I \\ \frac{1}{I}\int_{t-I}^{t} d\lambda & t>I \end{cases}$$

$$= \frac{1}{I}\left[t\left[u(t) - u(t-I)\right] + I\,u(t-I)\right]$$

$$= \frac{1}{I}\left[t\,u(t) - (t-I)\,u(t-I)\right]$$

(c) Find the RAMP response let $x(t) = t\,u(t)$

$$y(t) = \frac{1}{I}\int_{t-I}^{t} \lambda u(\lambda)\,d\lambda = \begin{cases} 0 & t<0 \\ \frac{1}{I}\int_{0}^{t} \lambda\,d\lambda & 0 \le t < I \\ \frac{1}{I}\int_{t-I}^{t} \lambda\,d\lambda & t \ge I \end{cases}$$

$$= \frac{1}{I}\left[\frac{t^2}{2}\left[u(t) - u(t-I)\right] + \left(\frac{t^2}{2} - \frac{(t-I)^2}{2}\right) u(t-I)\right]$$

$$= \frac{1}{I}\left[\frac{t^2}{2}\,u(t) - \frac{(t-I)^2}{2}\,u(t-I)\right]$$

(d) Show that for ramp input in steady-state $t > I$ the delay is $\frac{I}{2}$

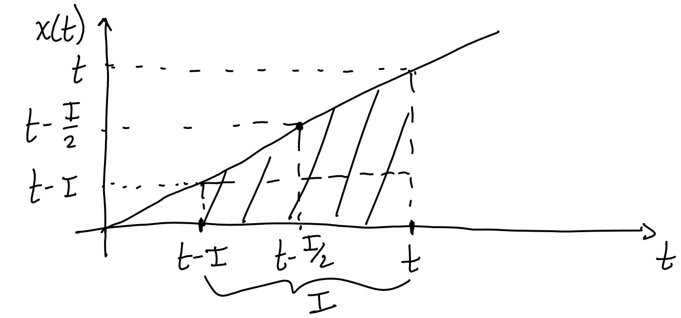
for $x(t) = r(t)$ and $t > I$

$$y(t) = \frac{1}{I}\left[\frac{t^2}{2} - \frac{(t-I)^2}{2}\right] = \frac{1}{2I}\left[t^2 - (t^2 - 2tI + I^2)\right]$$

$$= \frac{1}{2I}\left[2tI - I^2\right] = \frac{1}{2}(2t - I) = \boxed{\left(t - \frac{I}{2}\right)}$$

Compare this to $x(t) = t$ for $t \geq 0$, $y(t)$ essentially has a delay of $\frac{I}{2}$

Graphically, this can be shown as



$$\text{mean} = \frac{\text{Area}}{I}$$

$$\frac{\text{Area}}{I} = \frac{I\,r(t-I) + \frac{1}{2}I\left[r(t) - r(t-I)\right]}{\cancel{I}} = (t-I) + \frac{1}{2}\left[t - t + I\right]$$

$$= t - \frac{I}{2}$$

In other words, the moving average at time "t" is equal to the value of the ramp function at $t - \frac{I}{2}$ or $r\left(t - \frac{I}{2}\right)$

Problem 3

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Consider the following two systems. Determine the impulse response of entire system

$$x(t) \longrightarrow \boxed{h_1(t)} \xrightarrow{v(t)} \boxed{h_2(t)} \longrightarrow y(t)$$

For all cases: $v(t) = x(t) * h_1(t)$

$$y(t) = v(t) * h_2(t) = x(t) * \underbrace{\left[h_1(t) * h_2(t)\right]}_{H(t)} = x(t) * H(t)$$

$$H(t) = h_1(t) * h_2(t)$$

(a) $h_1(t) = \delta(t) \quad h_2(t) = 2e^{-t}u(t)$

$$H(t) = \delta(t) * 2e^{-t}u(t) = \int_{-\infty}^{\infty} \delta(\lambda)\, 2e^{-(t-\lambda)} u(t-\lambda)\, d\lambda$$

$$= 2e^{-t}u(t) \int_{-\infty}^{\infty} \delta(\lambda)\, d\lambda = 2e^{-t}u(t)$$

(b) $h_1(t) = e^{-t}u(t) \quad h_2(t) = 2\delta(t-1)$

$$H(t) = h_1(t) * h_2(t) = \int_{-\infty}^{\infty} 2\delta(\lambda-1)\, e^{-(t-\lambda)} u(t-\lambda)\, d\lambda$$

$$= 2e^{-(t-1)}u(t-1) \int_{-\infty}^{\infty} \delta(\lambda-1)\, d\lambda = 2e^{-(t-1)}u(t-1)$$

(c) $h_1(t) = 2\delta(t-1) \quad h_2(t) = 3\delta(t-2)$

$$H(t) = \int_{-\infty}^{\infty} 2\delta(\lambda-1)\, 3\delta(t-\lambda-2)\, d\lambda = 6\,\delta(t-3) \int_{-\infty}^{\infty} \delta(\lambda-1)\, d\lambda$$

$$= 6\delta(t-3)$$

(d) Any function $x(t)$ convolved with an impulse $A\delta(t-t_0)$ becomes $Ax(t-t_0)$ i.e. $x(t) * A\delta(t-t_0) = Ax(t-t_0)$

Problem 4

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Consider an LTI system with

$$h(t) = e^{-(t+1)} u(t+1) \quad \text{and} \quad x(t) = e^{-(t-1)} u(t-1)$$

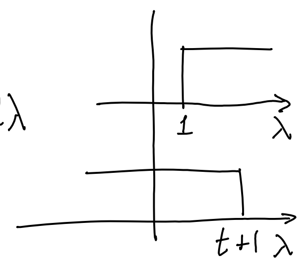
impulse response — input

(a) Analytical Convolution

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} e^{-(\lambda-1)} u(\lambda-1)\, e^{-(t-\lambda+1)} u(t-\lambda+1)\, d\lambda$$

$$= \int_{-\infty}^{\infty} e^{-t+\lambda-1-\lambda+1}\, u(\lambda-1)\, u(t-\lambda+1)\, d\lambda$$

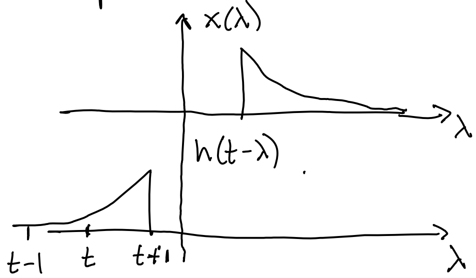
$$= e^{-t} \int_{-\infty}^{\infty} u(\lambda-1)\, u(-(\lambda-(t+1)))\, d\lambda$$



$$= \begin{cases} 0 & t<0 \\ e^{-t} \int_{1}^{t+1} d\lambda & t \geq 0 \end{cases}$$

$$\boxed{= t e^{-t} u(t)}$$

(b) Graphical Convolution



$$y(t) = \begin{cases} 0 & t < 0 \\ \displaystyle\int_1^{t+1} x(\lambda) h(t-\lambda)\, d\lambda & t \geq 0 \end{cases}$$

$$\int_1^{t+1} e^{-(\lambda-1)} e^{-(t-\lambda+1)}\, d\lambda = \int_1^{t+1} e^{-\lambda+1-t+\lambda-1}\, d\lambda = e^{-t} \int_1^{t+1} d\lambda = t e^{-t}$$

$$y(t) = \begin{cases} 0 & t < 0 \\ t e^{-t} & t \geq 0 \end{cases} = \boxed{t e^{-t} u(t)}$$

Problem 5

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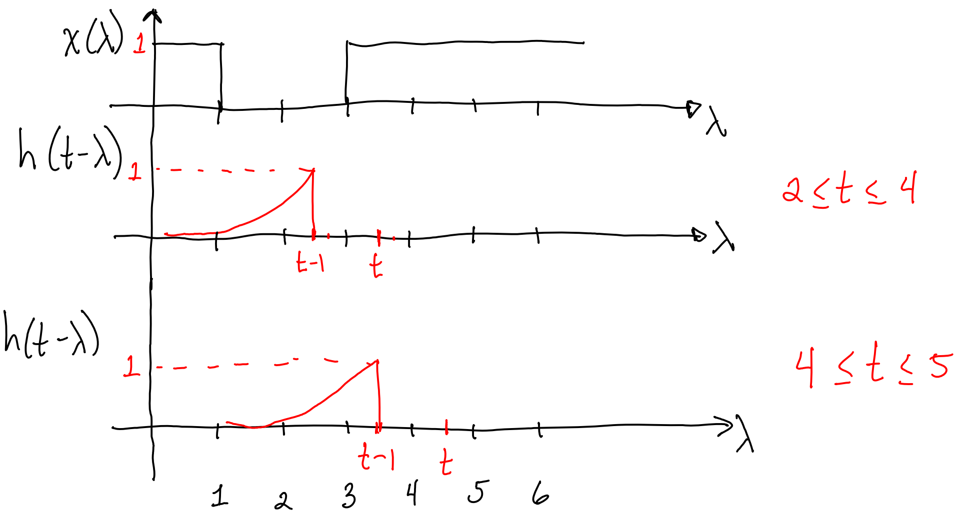
Given : $h(t) = e^{-(t-1)} u(t-1)$

$x(t) = u(t) - u(t-1) + u(t-3)$



for $2 \le t \le 4$

$$y(t) = \int_0^1 x(\lambda) h(t-\lambda) d\lambda = \int_0^1 e^{-(t-\lambda-1)} d\lambda = e^{-(t-1)} \int_0^1 e^{\lambda} d\lambda$$

$$\boxed{= e^{-(t-1)} \left[e^{1} - 1 \right]}$$

for $4 \le t \le 5$

$$y(t) = \int_0^1 x(\lambda) h(t-\lambda) d\lambda + e^{-(t-1)} \int_3^{t-1} e^{\lambda} d\lambda$$

$$\boxed{= e^{-(t-1)} \left[e^{1} - 1 \right] + e^{-(t-1)} \left[e^{t-1} - e^{3} \right]}$$

Problem 6

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Given : $h(t) = e^{-(t-1)} u(t-1)$

(a) Show that $s(t) = [1 - e^{-(t-1)}] u(t-1)$

$$s(t) = \int_{-\infty}^{t} h(t)\,dt = \begin{cases} 0 & t < 1 \\ e^{1}\int_{1}^{t} e^{-\lambda}\, d\lambda & t \geq 1 \end{cases}$$

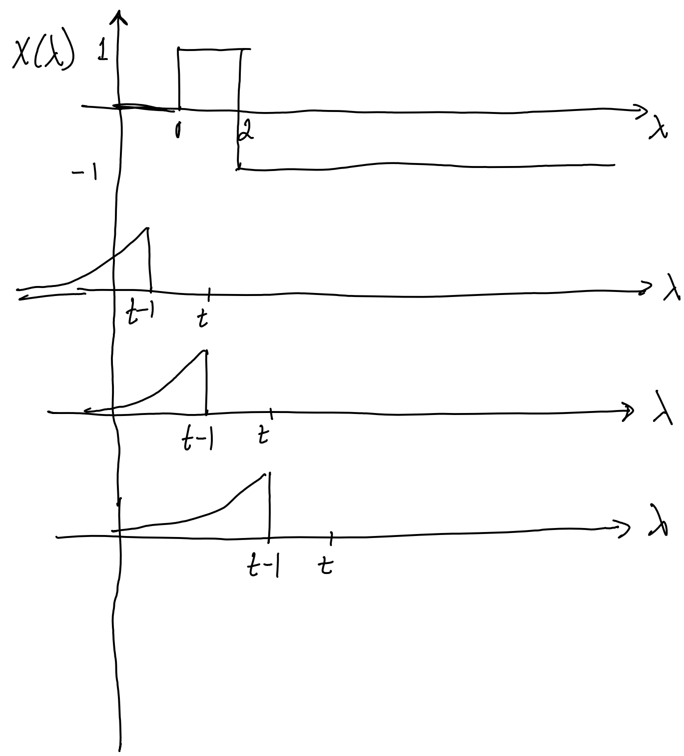
$$= -e^{1}\left[e^{-t} - e^{-1}\right] u(t-1) = \left[1 - e^{-(t-1)}\right] u(t-1)$$

(b) if $x(t) = u(t-1) - 2u(t-2)$

$$y(t) = s(t-1) - 2s(t-2) = \left[1 - e^{-(t-2)}\right]u(t-2) - 2\left[1 - e^{-(t-3)}\right] u(t-3)$$

$$= u(t-2) - 2u(t-3) - e^{-(t-2)}u(t-2) + 2e^{-(t-3)}u(t-3)$$

(c) Use graphical convolution



$h(t-\lambda)$ for $t \leq 2$

$h(t-\lambda)$ for $2 \leq t \leq 3$

$3 \leq t$

for $t < 2$

$$y(t) = 0$$

for $2 \le t \le 3$

$$y(t) = \int_1^{t-1} e^{-(t-\lambda-1)}\, d\lambda = e^{-(t-1)} \int_1^{t-1} e^{\lambda} d\lambda = e^{-(t-1)}\left[e^{(t-1)} - e^{1}\right]$$

$$= 1 - e^{-t+1+1} = 1 - e^{-(t-2)}$$

for $3 \le t$

$$y(t) = e^{-(t-1)} \int_1^{2} e^{\lambda} d\lambda - e^{-(t-1)} \int_2^{t-1} e^{\lambda} d\lambda$$

$$= e^{-(t-1)}\left[e^{2} - e^{1}\right] - e^{-(t-1)}\left[e^{(t-1)} - e^{2}\right]$$

$$= e^{-t+1+2} - e^{-t+1+1} - 1 + e^{-t+1+2}$$

$$= 2e^{-(t-3)} - e^{-(t-2)} - 1$$

Putting it all together:

$$y(t) = \left(1 - e^{-(t-2)}\right)\left[u(t-2) - u(t-3)\right] + \left(2e^{-(t-3)} - e^{-(t-2)} - 1\right) u(t-3)$$

$$= u(t-2) - u(t-3) - e^{-(t-2)}u(t-2) + e^{-(t-2)}u(t-3) + \left[2e^{-(t-3)} - e^{-(t-2)} - 1\right]u(t-3)$$

$$\boxed{= u(t-2) - 2u(t-3) - e^{-(t-2)}u(t-2) + 2e^{-(t-3)}u(t-3)}$$

(d) The answers to parts (b) and (c) are the same

(e) $s(t) = \left[1 - e^{-(t-1)}\right] u(t-1)$

$$\dot{s}(t) = \delta(t-1) - \delta(t-1)e^{-(t-1)} - u(t-1)\, e^{-(t-1)}(-1)$$

$$= \delta(t-1)\left[1 - e^{-(t-1)}\right] + e^{-(t-1)}u(t-1)$$

$$= \delta(t-1)\left[1 - e^{-(1-1)}\right] + e^{-(t-1)}u(t-1) = \boxed{e^{-(t-1)}u(t-1)}$$

Pre lab 4

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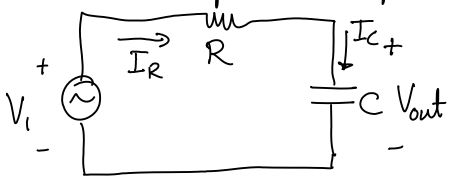
(a) Calculate the impulse response of RC lowpass circuit.



$I_R = I_C$

$$C\,\dot{V}_{out} = \frac{V_1 - V_{out}}{R}$$

$$\dot{V}_{out} + \frac{1}{RC} V_{out} = \frac{V_1}{RC} \longrightarrow$$

This is same as generic DE

$$\dot{y} + a y = b x$$

$$\boxed{h(t) = \frac{1}{RC} e^{-t/RC} u(t)}$$

$$h(t) = b e^{-at} u(t)$$

(b) The step response is given by:

$$s(t) = \int_{-\infty}^{t} h(\lambda)\, d\lambda = \frac{1}{RC}\int_{-\infty}^{t} e^{-\lambda/RC} u(\lambda)\, d\lambda = \begin{cases} 0 & \text{for } t<0 \\ \frac{1}{RC}\int_0^t e^{-\lambda/RC} d\lambda & \text{for } t \geq 0 \end{cases}$$

$$= -\left[e^{-t/RC} - 1\right]u(t) = \left(1 - e^{-t/RC}\right) u(t)$$

$$s(t_{90}) = 0.9 = \left(1 - e^{-t_{90}/\tau}\right) u(t_{90})$$

$$t_{90} = -\tau \ln(0.1)$$

$$s(t_{10}) = 0.1 = \left(1 - e^{-t_{10}/\tau}\right) u(t_{10})$$

$$t_{10} = -\tau \ln(0.9)$$

$$\text{Rise time} = t_{90} - t_{10} = \tau[\ln(0.9) - \ln(0.1)] = \boxed{\tau \ln(9)}$$

(c) If $\tau = RC = 1\,\text{msec}$ then $R = 10\,k\Omega$

$C = 0.1\,\mu F$

(d) Show that the response to the circuit of pulse with duration T and amplitude A is

$$y_{pulse}(t) = A\left[(1-e^{-t/\tau})u(t) - (1-e^{-(t-T)/\tau})u(t-T)\right]$$

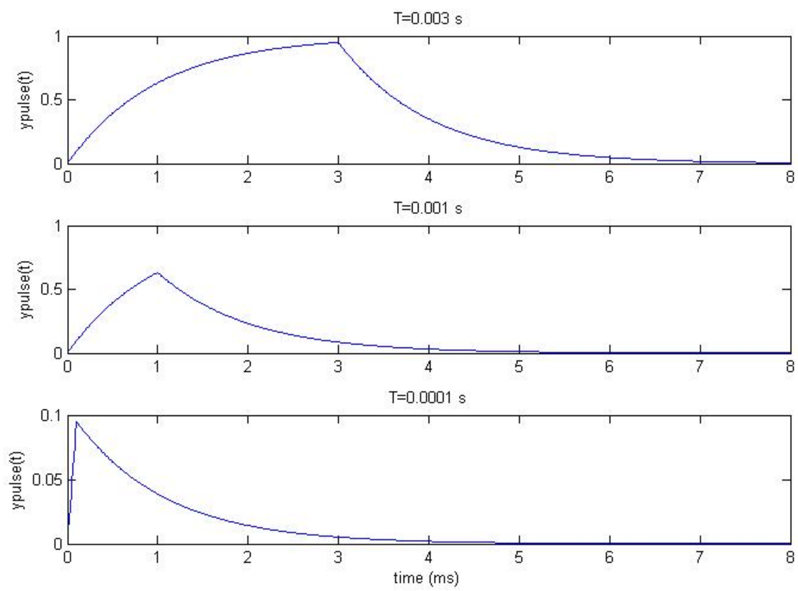
Answer: The pulse is given by $A[u(t)-u(t-T)]$

Thus $y_{pulse}(t)$ can be found by a combination of step responses of the system.

$$y_{pulse}(t) = A[s(t)-s(t-T)]$$

(e) plot $y_{pulse}(t)$ for * $A=1$ & $\tau = 0.001$ seconds

* $T = 0.003$, 0.001, and 0.0001 seconds



(f) find $y_{pulse}(T)$

$$y_{pulse}(T) = A\left[(1-e^{-T/\tau})u(T) - \underbrace{\cancel{(1-e^{-(T-T)/\tau})}}_{\to 0}u(T-T)\right]$$

$$= A(1-e^{-T/\tau})$$

The Taylor series expansion of $e^{x} = 1 + x$

$$y_{pulse}(T) \approx A(1-(1-T/\tau)) = \frac{AT}{\tau} = \frac{Area}{\tau}$$