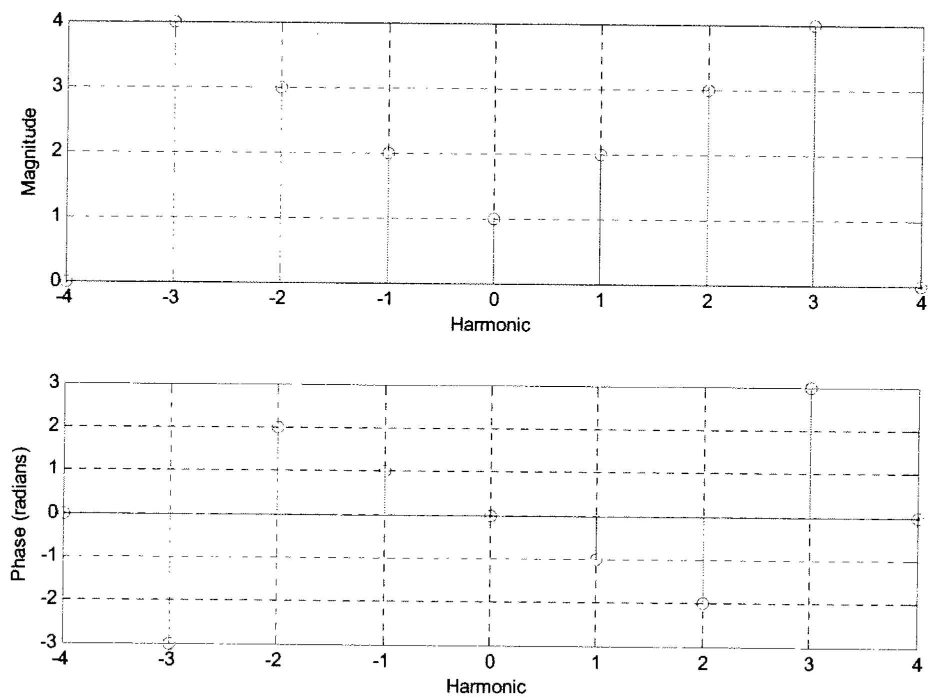
**ECE 300**
**Signals and Systems**
Homework 6

**Due Date:** Tuesday January 22, 2008 at the beginning of class

**Exam 2, Thursday January 24, 2008**

**Problems:**

**1.** Assume $x(t)$ has the spectrum shown below (the phase is shown in radians) and a fundamental frequency $\omega_o = 2 \text{ rad/sec}$:




Assume $x(t)$ is the input to a system with the transfer function

$$H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| < 5 \\ 0 & else \end{cases}$$

Determine an expression for the steady state output $y(t)$. Be as specific as possible, simplifying all values and using actual numbers wherever possible.

**2.** ZTM, Problem 3-16.

**3.** A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal highpass filter that eliminates all signals with frequency content less than 0.75 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real.

(*Answer:* $y(t) = e^{-t} - 0.4323 - 0.2622\cos(\pi t - 1.2626)$ )

c) Determine the average power in $y(t)$.

d) What fraction of the average power in $x(t)$ is contained in the DC and fundamental frequency components?

**4.** Assume $x(t)$ has the Fourier series representation $x(t) = \sum X_k e^{jk\omega_o t}$ and fundamental period $T_o$. The function $y(t)$ is related to $x(t)$ through the relationship $y(t) = x\left(\frac{t}{b}\right)$.

a) Determine the period of $y(t)$ in terms of $T_o$ (the period of $x(t)$) and fundamental frequency for $y(t)$ in terms of $\omega_o$ (the fundamental frequency for $x(t)$)

b) Set up the integral to determine the Fourier series coefficients $Y_k$ in terms of the parameters determined in part a (the integral should be centered at 0), and determine how $Y_k$ is related to $X_k$

c) Starting from the relationship $x(t) = \sum X_k e^{jk\omega_o t}$ and making a simple substitution, show how we can determine the results from part b.

*This problem demonstrates that compression or expansion of a signal does not change the Fourier series coefficients, it only changes the fundamental frequency.*

The Fourier series representation is $v(t)=\sum_{k=-\infty}^{k=\infty} e^{-\frac{jk\pi\tau}{T_0}}\left(\frac{V\tau}{T_0}\right)\mathrm{sinc}\left(\frac{k\tau}{T_0}\right)e^{jk\omega_0 t}$

**a)** Determine an expression for the **average value** of $v(t)$ in terms of $T_0$, $\tau$, and $V$.

**b)** It is the average value of $v(t)$ that we want to use as our analog output. Hence we need to design a lowpass filter that allows us to keep our DC term, and, ideally, remove all of the other harmonics. Let's assume we want to use a simple first order RC lowpass filter with transfer function

$$H(j\omega)=\frac{1}{j\omega RC+1}=\frac{1}{j\omega\frac{\sqrt{\alpha}}{\omega_0}+1}$$

where we have set $RC=\frac{\sqrt{\alpha}}{\omega_0}$ for convenience.

Determine the value of $\alpha$ so that the average power in the first harmonic of the output signal is 20 dB lower than the average power in the DC component the output signal. Assume here that the fundamental frequency is $f_0=100\,Hz$, the duty cycle is $\frac{\tau}{T_0}=0.8$, and $V=5.0$ volts.

**c)** For your value of $\alpha$ determined in part **b** and the parameter values given in part **b**, determine an expression for the first two terms (the DC and first harmonic) in the Fourier series representation of the output signal.
(Answer: $y(t)\approx 4+0.566\cos(2\pi 100t-3.77)$ )

**7.** Review of Logarithms and dB. _Read the documents available online (website, ANGEL, etc) to answer these questions._

**a)** Express the value of -120dBm in terms of dBmV

**b)** Express the value of 60dBmV in terms of the amplitude of a cosine function.

**c)** Express the average power in the signal $x(t)=4\cos(100\pi t-1.21)$ V in terms of dB

**d)** If the average power of a sinusoidal signal into a 1Ω resistor is increased by 12dBm by what factor is the amplitude of the sinusoidal signal increased.

**5.** Assume two periodic signals have the Fourier series representations

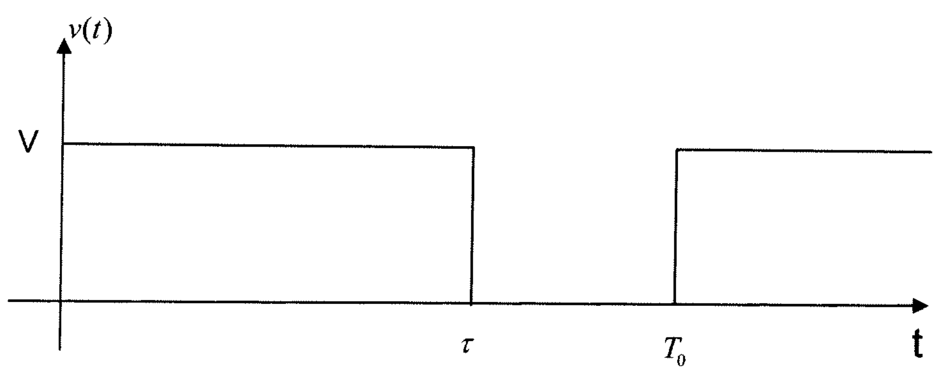
$$x(t)=\sum X_k e^{jk\omega_o t} \quad y(t)=\sum Y_k e^{jk\omega_o t}$$

For the following system (input/output) relationships:

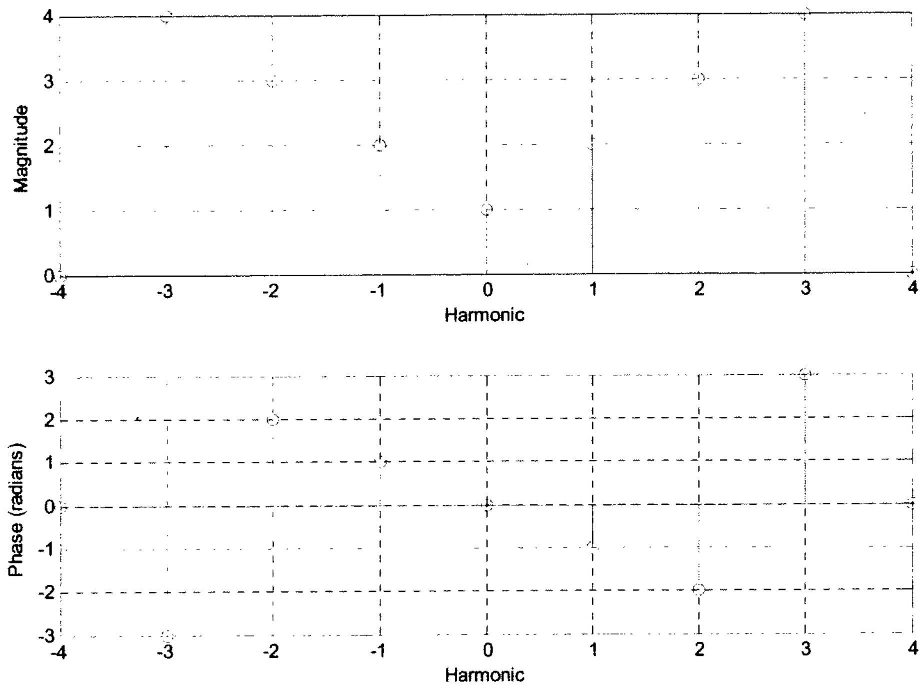
a) $y(t)=bx(t-a)$
b) $y(t)=b\dot{x}(t-a)$
c) $y(t)=bx(t)\cos(\omega_o t)$ (Answer: $Y_n=\frac{b}{2}(X_{n-1}+X_{n+1})$)
d) $\ddot{y}(t)+\frac{2\varsigma}{\omega_n}\dot{y}(t)+\frac{1}{\omega_n^2}y(t)=Kx(t)$

i) write $Y_k$ in terms of the $X_k$
ii) If possible, determine the system transfer function $H(j\omega)$
iii) A system must be both linear and time-invariant to have a transfer function. If you cannot determine the transfer function, indicate which system property is not satisfied (**L** or **TI**).

**6.** Most microcontrollers are capable of generating pulse width modulation (PWM) signals on one or more output pins. These signals are square waves where both the period and the duty cycle can be programmed in the microcontroller by the use of timers and different reference clocks. These PWM output signals can then be used in conjunction with lowpass filters to produce reasonable approximations to analog output signals. In this problem we will use what we have learned in the course to investigate how to do this. The signal $v(t)$ below is a PWM signal, shown for about one and a half periods. The signal has period $T_0$, amplitude $V$ (usually fixed at 5 or 3.3 volts), pulse duration $\tau$, and duty cycle $D=\frac{\tau}{T_0}$.



#1 $H(\omega) = \begin{cases} e^{-j\omega} & 1 \le |\omega| < 3 \\ 2e^{-j2\omega} & 3 < |\omega| 5 \\ 0 & \text{else} \end{cases}$ $\omega_0 = 2\text{rad/sec}$




$$Y_0 = X_0 H(0) = 0$$

$$Y_1 = X_1 H(1\omega_0) = \left(2e^{-j1}\right)\left(e^{-j2}\right) = 2e^{-j3} = 2 \angle -3 \text{ rad}$$

$$Y_2 = X_2 H(2\omega_0) = \left(3e^{-j2}\right)\left(2e^{-j8}\right) = 6e^{-j10} = 2 \angle -10 \text{ rad}$$

$$Y_3 = X_3 H(3\omega_0) = 0$$

$$y(t) = Y_0 + 2|Y_1|\cos(\omega_0 t + \angle Y_1) + 2|Y_2|\cos(2\omega_0 t + \angle Y_2) + 0 + \cdots$$

$$\boxed{y(t) = 4\cos(2t-3) + 12\cos(4t-10)}$$

#2 ZTF Problem 3-16 $\quad x(t) = \sum x_n e^{jn\omega_0 t} \quad X_n = \frac{1}{T_0}\int_{T_0} x(t) e^{-jn\omega_0 t} dt$

a) assume $x(t)$ is real and even

$$X_n = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\left[\cos(n\omega_0 t) - j\sin(n\omega_0 t)\right] dt$$

$$= \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x(t)\cos(n\omega_0 t) dt - j\,\underbrace{\frac{1}{T_0}\int_{-\frac{T_0}{2}}^{T_0/2} \underbrace{\underbrace{x(t)}_{\text{even}}\,\underbrace{\sin(n\omega_0 t)}_{\text{odd}}}_{\text{product is odd}} dt}_{\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} \text{odd function} = 0}$$

so $X_n = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(n\omega_0 t) dt$

$$X_{-n} = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\cos(-n\omega_0 t) dt = X_n \quad \text{(cosine is even)}$$

since $x(t)$ is real and even we have $\boxed{X_n \text{ is real and even}}$

b) assume $x(t)$ is real and odd

Then by the above arguments, $X_n = \frac{-j}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(n\omega_0 t) dt$

$$X_{-n} = \frac{-j}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(-n\omega_0 t) dt = \frac{j}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t)\sin(n\omega_0 t) dt = -X_n$$

since $x(t)$ is real and odd, $\boxed{X_n \text{ is purely imaginary and odd}}$

c) assume $x(t) = -x(t \pm \frac{T_0}{2})$ (half wave symmetry)

$$x(t) = \sum X_n e^{jn\omega_0 t} = -x(t+\tfrac{T_0}{2}) = -\sum X_n e^{jn\omega_0(t+T_0/2)}$$

$$= \underbrace{-}\sum X_n e^{jn\omega_0 t} e^{jn(\omega_0 T_0)/2} = \underbrace{e^{j\pi}}\sum X_n e^{jn\omega_0 t} e^{jn\pi}$$

$$= \sum X_n e^{jn\omega_0 t} e^{j(n+1)\pi} = \sum X_n e^{jn\omega_0 t} = x(t)$$

This will only happen if $X_n = 0$ for $n$ even

#3 $x(t) = e^{-t} \quad 0 \le t \le 2$ $\qquad T_0 = 2 \quad f_0 = \frac{1}{2} = 0.5\,\text{Hz}$

$$x(t) = \sum_{k=-\infty}^{\infty} \frac{0.4323}{1+jk\pi} e^{jk\pi t}$$

a) $$P_{ave}^{x} = \frac{1}{T_0}\int_{T_0} |x(t)|^2 dt = \frac{1}{2}\int_0^2 e^{-2t} dt = \frac{1}{2} \left. \frac{e^{-2t}}{-2} \right|_0^2 = \frac{1}{4}\left(1-e^{-4}\right) = 0.2454$$

$$\boxed{P_{ave}^{x} = 0.2454}$$

b) The high pass filter removes signals with frequency content below 0.75 Hz. Let's figure out what they are

It removes $k=0$, $k=\pm 1$

$c_0^x = 0.4323$

$c_1^x = \frac{0.4323}{1+j\pi} = 0.13112 \measuredangle 1.2626$ rad

$y(t) = e^{-t} - 0.4323 - 2(0.13112)\cos(\pi t - 1.2626)$

$$\boxed{y(t) = e^{-t} - 0.4323 - 0.26225\cos(\pi t - 1.2626)}$$

c) $P_{ave}^{y} = P_{ave}^{x} - |c_0^x|^2 - 2|c_1^x|^2 = 0.2454 - (0.4323)^2 - 2(0.13112)^2$

$$\boxed{P_{ave}^{y} = 0.02413}$$

d) $$\frac{|c_0^x|^2 + 2|c_1^x|^2}{P_x^{ave}} = \frac{(0.4323)^2 + 2(0.13112)^2}{0.2454} = 0.90166 \approx 90\%$$

#4 $x(t) = \sum x_K e^{jK\omega_0 t}$ $\quad X_K = \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(t) e^{-jK\omega_0 t} dt$

$y(t) = x\left(\frac{t}{b}\right)$

a) $y(bT_0) = x(T_0)$ so the period of $y(t)$ is $bT_0$ and the fundamental frequency is

$$\frac{2\pi}{bT_0} = \frac{\omega_0}{b}$$

b) $$Y_n = \frac{1}{bT_0}\int_{-\frac{bT_0}{2}}^{\frac{bT_0}{2}} x\left(\frac{t}{b}\right) e^{-jK\frac{\omega_0}{b}t} dt$$

let $\sigma = \frac{t}{b}$ $\quad b\sigma = t$ $\quad b\,d\sigma = dt$

$$Y_n = \frac{1}{bT_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(\sigma) e^{-jK\frac{\omega_0}{b}b\sigma} b\,d\sigma$$

$$= \frac{1}{T_0}\int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} x(\sigma) e^{-jK\omega_0\sigma} d\sigma = X_n$$

so $\boxed{Y_n = X_n}$

c)
$x(t) = \sum X_K e^{jK\omega_0 t}$

$y(t) = x\left(\frac{t}{b}\right) = \sum X_K e^{jK\omega_0\frac{t}{b}} = \sum x_K e^{jK\left(\frac{\omega_0}{b}\right)t}$

so $Y_K = X_K$

#5 $\quad x(t) = \sum X_K e^{jK\omega_0 t} \qquad y(t) = \sum Y_K e^{jK\omega_0 t}$

a) $y(t) = b\,x(t-a)$

$$\sum Y_K e^{jK\omega_0 t} = b \sum X_K e^{jK\omega_0 (t-a)} = \sum b X_K e^{-jK\omega_0 a} e^{jK\omega_0 t}$$

$$\boxed{Y_K = X_K b e^{-jK\omega_0 a} \qquad H(j\omega) = b e^{-j\omega a}}$$

b) $y(t) = b\,\dot{x}(t-a)$

$$\sum Y_K e^{jK\omega_0 t} = \frac{d}{dt} \sum b e^{-jK\omega_0 a} X_K e^{jK\omega_0 t}$$

$$= \sum b e^{-jK\omega_0 a} jK\omega_0 X_K e^{jK\omega_0 t}$$

$$\boxed{Y_K = b e^{-jK\omega_0 a} jK\omega_0 X_K \qquad H(j\omega) = b j\omega e^{-j\omega a}}$$

c) $y(t) = b\,x(t)\cos(\omega_0 t)$

$$Y_K = \frac{1}{T_0} \int_{T_0} y(t) e^{-jK\omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{T_0} b\,x(t) \cos(\omega_0 t) e^{-jK\omega_0 t} dt$$

$$= \frac{1}{T_0} \int_{T_0} b\,x(t) \left[\frac{e^{j\omega_0 t}}{2} + \frac{e^{-j\omega_0 t}}{2}\right] e^{-jK\omega_0 t} dt$$

$$= \frac{b}{2} \left[\frac{1}{T_0} \int_{T_0} x(t) e^{-j(K-1)\omega_0 t} dt + \frac{1}{T_0} \int_{T_0} x(t) e^{-j(K+1)\omega_0 t} dt\right]$$

$$\boxed{Y_K = \frac{b}{2}\left[X_{K-1} + X_{K+1}\right]}$$

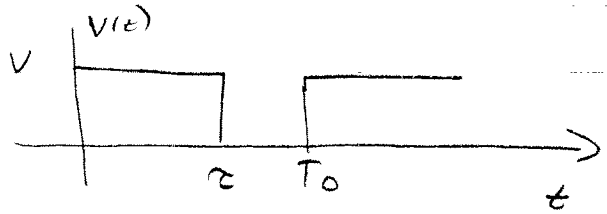
not TI

d) $\ddot{y}(t) + \frac{2\zeta}{\omega_n}\dot{y}(t) + \frac{1}{\omega_n^2} y(t) = K x(t)$

$$\sum Y_K (jK\omega_0)^2 e^{jK\omega_0 t} + \sum Y_K \frac{2\zeta}{\omega_n}(jK\omega_0) e^{jK\omega_0 t} + \sum Y_K \frac{1}{\omega_n^2} e^{jK\omega_0 t} = \sum X_n K e^{jK\omega_0 t}$$

$$\boxed{Y_K = \frac{K}{(jK\omega_0)^2 + \frac{2\zeta}{\omega_n}(jK\omega_0) + \frac{1}{\omega_n^2}} X_K \qquad H(j\omega) = \frac{K}{(j\omega)^2 + \frac{2\zeta}{\omega_n}(j\omega) + \frac{1}{\omega_n^2}}}$$

#6




$$v(t) = \sum_{k=-\infty}^{\infty} e^{-jk\pi\tau/T_0} \left(\frac{V\tau}{T_0}\right) \operatorname{sinc}\left(k\frac{\tau}{T_0}\right) e^{jk\omega_0 t}$$

a) $$c_0 = \frac{1}{T_0}\int_0^{\tau} V\,dt = \boxed{\frac{V\tau}{T_0} = c_0}$$

b) $P_{ave}^{1} = 2|c_1^y|^2 \qquad P_{ave}^{0} = |c_0^y|^2$

$$-20\,dB = 10\log_{10}\left(\frac{P_{ave}^{1}}{P_{ave}^{2}}\right) = 10\log_{10}\left(\frac{2|c_1^y|^2}{|c_0^y|^2}\right)$$

$$-2 = \log_{10}\left(\frac{2|c_1^y|^2}{|c_0^y|^2}\right) \qquad 10^{-2} = 0.01 = \frac{2|c_1^y|^2}{|c_0^y|^2}$$

$$c_0^y = c_0^x H(0) = \frac{V\tau}{T_0}(1) = 5(0.8)(1) = 4$$

$$|c_1^y|^2 = 0.01\frac{|c_0^y|^2}{2} = 0.01\frac{(4)^2}{2} = 0.08$$

$$|c_1^y| = |c_1^x H(j\omega_0)| = |c_1^x||H(j\omega_0)|$$

$$= \left|\frac{V\tau}{T_0}\operatorname{sinc}\left(\frac{\tau}{T_0}\right)e^{-j\omega_0\tau/T_0}\right|\left|\frac{1}{1+j\sqrt{\alpha}}\right|$$

$$= \frac{V\tau}{T_0}\left|\operatorname{sinc}\left(\frac{\tau}{T_0}\right)\right|\frac{1}{\sqrt{1+\alpha}}$$

$$= \frac{4\operatorname{sinc}(0.8)}{\sqrt{1+\alpha}} \qquad \operatorname{sinc}(0.8) = \frac{\sin(\pi\cdot 0.8)}{\pi\cdot 0.8}$$

$$= 0.234$$

$$= \frac{0.9355}{\sqrt{1+\alpha}} = \sqrt{0.08}$$

$$\frac{0.955}{\sqrt{0.08}} = \sqrt{1+\alpha} \qquad \alpha = 9.94$$

#6 (continued)

c) $c_0^y = 4 \qquad c_1^y = c_1^x H(j\omega_0)$

$$= \frac{V\tau}{T_0} \operatorname{sinc}\left(\frac{\tau}{T_0}\right) e^{-j\pi\tau/T_0} \frac{1}{1+j\sqrt{a}}$$

$$= (4)(0.234)\, e^{-j0.8\pi} \frac{1}{1+j\sqrt{9.94}}$$

$$= 0.283\, e^{-j3.78}$$

$$y(t) = c_0^x + 2|c_1^y| \cos(\omega_0 t + \measuredangle c_1^y)$$

$$\boxed{y(t) = 4 + 0.566 \cos(2\pi \cdot 100 t - 3.78)}$$

check $\quad 10\log_{10}\left(\dfrac{4^2}{\frac{1}{2}(0.566)}\right) = 19.995 \text{ dB}$

# Problem 7

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a) Express $-120$ dBm in dBmV

$$-120\ \text{dBm} = 10\log_{10}\left(\frac{V_{rms}^2}{0.001}\right)$$

$$V_{rms}^2 = (10^{-12})(0.001) = 1\times10^{-15}$$

$$V_{rms} = \sqrt{1\times10^{-15}}$$

Convert to dBmV

$$= 10\log_{10}\left(\frac{V_{rms}^2}{(0.001)^2}\right) = -90\ \text{dBmV}$$

b) Express 60 dBmV in terms of cosine Amplitude

$$60\ \text{dBmV} = 10\log_{10}\left(\frac{V_{rms}^2}{(0.001)^2}\right)$$

$$V_{rms} = 10^6(0.001)^2 = 1 = \frac{V_p}{\sqrt{2}}$$

$$\boxed{V_p = \sqrt{2}}$$

(c) Express the average Power in signal $x(t) = 4\cos(100\pi t - 1.21)$ in terms of dB.

$$P_{ave} = 10\log_{10}\left(\frac{V_{rms}^2}{1\Omega}\right) = 20\log_{10}\left(V_{rms}\right)$$

$$= 20\log_{10}\left(\frac{4}{\sqrt{2}}\right) = 9.03\ \text{dB}$$

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d) $P_{ave} + 12\ dBm = 10 \log_{10}\left(\frac{V_{rms}^2}{0.001}\right) + 10 \log_{10}(15.84)$

$$\boxed{\begin{aligned} 12dBm &= 10 \log_{10}(x) \\ x &= 10^{1.2} = 15.84 \end{aligned}}$$

$$P_{ave} + 12dBm = 10 \log_{10}\left(\frac{15.84\ V_{rms}^2}{0.001}\right)$$

$$= 10 \log_{10}\left(\frac{(3.98\ V_{rms})^2}{0.001}\right)$$

$$3.98\ V_{rms} = 3.98\ \frac{V_p}{\sqrt{2}}$$

So the peak amplitude has been increased by a factor of approx. 4.