ECE 300
Signals and Systems
Homework 3

Due Date: Tuesday December 18. 2007 *at the beginning of class*

EXAM #1, Thursday December 20

Problems

1. ZTF Problem 2-19 (use the method from class, solve the DE).

2. ZTF Problem 2-24.

3. The continuous-time I - interval moving average (MA) filter is given by the input/output relationship

$$y(t) = \frac{1}{I}\int_{t-I}^{t} x(\lambda)d\lambda$$

a. Determine the impulse response of the system. Write your answers in terms of unit step functions.

b. Determine the step response of the system, that is, determine the output when the input is a unit step. (*Answer:* $y(t) = \frac{1}{I}[tu(t) - (t-I)u(t-I)]$)

c. Determine the ramp response of the system, that is, determine the output when the input is a unit ramp.

d. Show that in steady state $(t > I)$ the delay between the input and output is $\frac{I}{2}$

Hint: Draw pictures of the integrand and look at what happens as the interval $[t, t-I]$ varies.

$$y(t) = \begin{cases} e^{-(t-1)}[e^1 - 1] & 2 \le t \le 4 \\ e^{-(t-1)}[e^1 - 1] + e^{-(t-1)}[e^{t-1} - e^3] & 4 \le t \le 5 \end{cases}$$

6. Consider a linear time invariant system with impulse response given by

$$h(t) = e^{-(t+1)}u(t+1)$$

The input to the system is given by

$$x(t) = 2[u(t) - u(t-1)] + 3[u(t-3) - u(t-4)]$$

Using ***graphical convolution***, determine the output $y(t)$. Specifically, you must

a) Flip and slide $h(t)$
b) Show graphs displaying both $h(t-\lambda)$ and $x(\lambda)$ for each region of interest
c) Determine the range of t for which each part of your solution is valid
d) Set up any necessary integrals to compute $y(t)$ ***Do Not Evaluate the Integrals***

7. Pre-Lab Exercises (to be done by all students. Turn this in with your homework and bring a copy of this with you to lab!)

a) Calculate the impulse response of the RC lowpass filter shown in Figure 2, in terms of unspecified components R and C. Determine the time constant for the circuit.

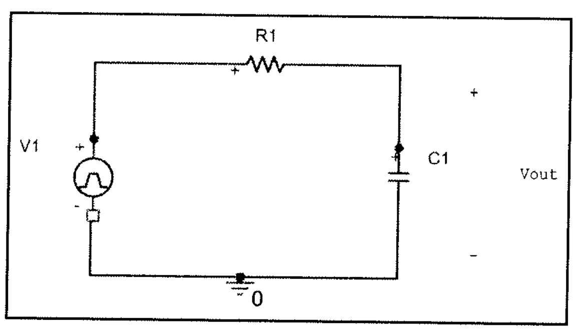
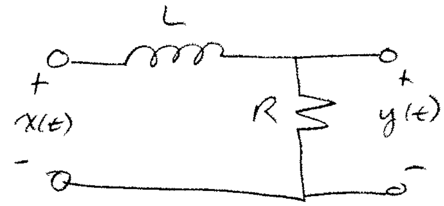


Figure 2. Simple RC lowpass filter circuit.

b) Find the **step response** of the circuit (the response of the system when the input is a unit step), and determine the 10-90% rise time. t_r, as shown below in Figure 3. The rise time is simply the amount of time necessary for the output to rise from 10% to 90% of its final value. Specifically, show that the rise time is given by $t_r = \tau \ln(9)$

ZTF Problem 2-19



$$V_L(t) = L\frac{di_L(t)}{dt}$$

$$V_L(t) = x(t) - y(t)$$

$$i_L(t) = \frac{y(t)}{R}$$

so $x(t) - y(t) = L\frac{\dot{y}(t)}{R}$

$$\frac{L}{R}\dot{y}(t) + y(t) = x(t)$$

$$\dot{y}(t) + \frac{R}{L}y(t) = \frac{R}{L}x(t)$$

for impulse response $x(t) \to \delta(t)$
$y(t) \to h(t)$
initial conditions are zero

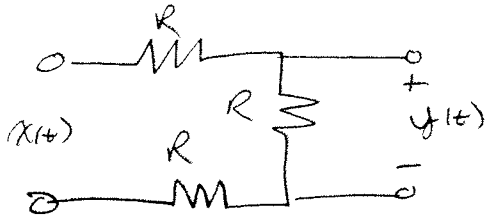
$$\dot{h}(t) + \frac{R}{L}h(t) = \frac{R}{L}\delta(t)$$

$$\frac{d}{dt}\left(h(t)e^{\frac{R}{L}t}\right) = e^{\frac{R}{L}t}\frac{R}{L}\delta(t)$$

$$\int_{-\infty}^{t}\frac{d}{d\lambda}\left(h(\lambda)e^{\frac{R}{L}\lambda}\right)d\lambda = h(t)e^{\frac{R}{L}t} = \int_{-\infty}^{t}e^{\frac{R}{L}\lambda}\frac{R}{L}\delta(\lambda)d\lambda = \frac{R}{L}u(t)$$

$$\boxed{h(t) = \frac{R}{L}e^{-\frac{R}{L}t}u(t)}$$

ZTF Problem 2-20.



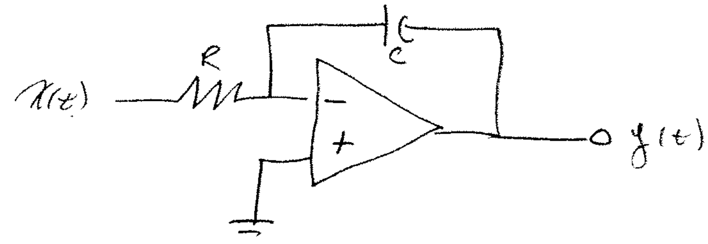
$$y(t) = x(t)\frac{R}{3R} = \frac{x(t)}{3}$$

impulse response $x(t) \rightarrow \delta(t)$

$y(t) \rightarrow h(t)$

$$\boxed{h(t) = \frac{1}{3}\delta(t)}$$

ZTF Problem 2-24



$$\frac{x(t)}{R} + C\frac{dV_C}{dt} = 0 \qquad V_C(t) = y(t)$$

$$C\frac{dy(t)}{dt} + \frac{x(t)}{R} = 0$$

$$\frac{dy(t)}{dt} + \frac{1}{RC}x(t) = 0$$

for impulse response $x(t) \rightarrow \delta(t)$

$y(t) \rightarrow h(t)$

initial conditions are zero

$$\dot{h}(t) + \frac{1}{RC}\delta(t) = 0$$

$$\frac{dh}{dt} = \frac{-1}{RC}\delta(t)$$

$$\int_{-\infty}^{t} \frac{dh(\lambda)}{d\lambda}d\lambda = \int_{-\infty}^{t} -\frac{1}{RC}\delta(\lambda)d\lambda$$

$$\boxed{h(t) = -\frac{1}{RC}u(t)}$$

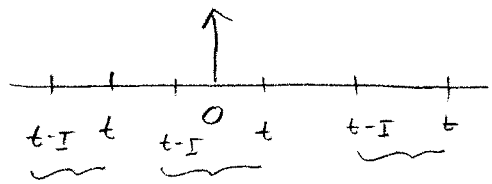
$$y(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t-\lambda)x(\lambda)d\lambda = \int_{-\infty}^{\infty} -\frac{1}{RC}u(t-\lambda)x(\lambda)d\lambda$$

$$\boxed{y(t) = -\frac{1}{RC}\int_{-\infty}^{t} x(\lambda)d\lambda}$$

#3 $\quad y(t) = \frac{1}{T}\int_{t-T}^{t} x(\lambda)\,d\lambda$

a) impulse response

$$h(t) = \frac{1}{T}\int_{t-T}^{t} \delta(\lambda)\,d\lambda$$



$$= \frac{1}{T} \text{ for } t>0 \text{ and } t-T<0$$

$$= \frac{1}{T} \text{ for } t>0 \text{ and } t<T$$

$$\boxed{h(t) = \frac{1}{T}\left[u(t) - u(t-T)\right]}$$

b) unit step response

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty} h(\lambda)u(t-\lambda)\,d\lambda$$

$$= \frac{1}{T}\int_{-\infty}^{\infty}\left[u(\lambda) - u(\lambda-T)\right]u(t-\lambda)\,d\lambda = \frac{1}{T}\int_{-\infty}^{\infty} u(\lambda)u(t-\lambda)\,d\lambda - \frac{1}{T}\int_{-\infty}^{\infty} u(\lambda-T)u(t-\lambda)\,d\lambda$$

$u(\lambda) = 1 \quad \lambda > 0$

$u(t-\lambda) = 1 \quad t-\lambda > 0 \quad t > \lambda$

$u(\lambda-T) = 1 \quad \lambda - T > 0 \quad \lambda > T$

need $t>0$ $\qquad$ need $t>T$

$$y(t) = \frac{1}{T}\left[\int_0^t d\lambda + \int_T^t d\lambda\right] = \frac{1}{T}\left[t\,u(t) + (t-T)\,u(t-T)\right]$$

$$\boxed{\text{step response} = \frac{1}{T}\left[t\,u(t) - (t-T)\,u(t-T)\right]}$$

c) ramp response

$$y(t) = h(t) * r(t) = \int_{-\infty}^{\infty} r(\lambda) h(t-\lambda) d\lambda$$

$$= \int_{-\infty}^{\infty} \lambda u(\lambda) \frac{1}{I} \left[u(t-\lambda) - u(t-I-\lambda) \right] d\lambda$$

$$= \frac{1}{I} \int_{-\infty}^{\infty} \lambda u(\lambda) u(t-\lambda) d\lambda - \frac{1}{I} \int_{-\infty}^{\infty} \lambda u(\lambda) u(t-I-\lambda) d\lambda$$

$$= \frac{1}{I} \int_{0}^{t} \lambda d\lambda - \frac{1}{I} \int_{0}^{t-I} \lambda d\lambda = \frac{1}{I} \frac{t^2}{2} u(t) - \frac{1}{I} \frac{(t-I)^2}{2} u(t-I)$$

$$\boxed{\text{ramp response} = \frac{1}{I} \left[\frac{t^2}{2} u(t) - \frac{(t-I)^2}{2} u(t-I) \right]}$$

d) for $t > I$ ramp response $= \frac{1}{I} \left[\frac{t^2}{2} - \frac{(t-I)^2}{2} \right]$

$$= \frac{1}{I} \left[\frac{t^2}{2} - \frac{(t^2 - 2tI + I^2)}{2} \right]$$

$$= \frac{1}{2I} \left[2tI - I^2 \right] = t - \frac{I}{2}$$

input $= t$ output $= t - I/2$ $\boxed{\text{delay} = I/2}$

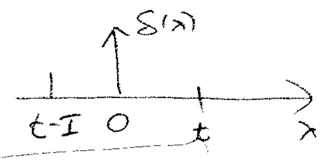
ECE 300 Homework #3

(#3) (Alternative Solution)

$$y(t) = \frac{1}{I}\int_{t-I}^{t} x(\lambda)\,d\lambda \quad I>0$$ continuous-time moving average filter

(a) assume $x(t) = \delta(t)$ so $y(t) = \frac{1}{I}\int_{t-I}^{t} \delta(\lambda)\,d\lambda$

draw a picture!

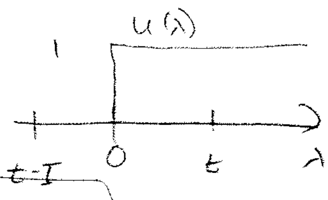


$y(t) = \frac{1}{I}$ if $\delta(\lambda)$ is in the region of integration

$$\boxed{y(t) = \begin{cases} \frac{1}{I} & 0<t<I \\ 0 & \text{else} \end{cases}} = \frac{1}{I}\left[u(t) - u(t-I)\right]$$

(b) assume $x(t) = u(t)$ so $y(t) = \frac{1}{I}\int_{t-I}^{t} u(\lambda)\,d\lambda$

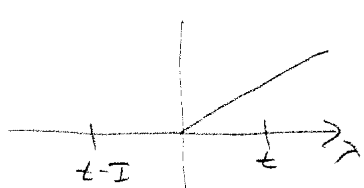
draw a picture!



$$\boxed{y(t) = \begin{cases} \frac{t}{I} & 0<t<I \\ 1 & t\geq I \\ 0 & \text{else} \end{cases}} = \frac{1}{I}\left[t\,u(t) - (t-I)\,u(t-I)\right]$$

(c) assume $x(t) = t\,u(t)$ so $y(t) = \frac{1}{I}\int_{t-I}^{t} \lambda\,u(\lambda)\,d\lambda$

draw a picture!



for $0<t<I$ $\quad y(t) = \frac{1}{I}\int_{0}^{t} \lambda^2\,d\lambda = \frac{t^2}{2I}$

for $t>I$ $\quad y(t) = \frac{1}{I}\int_{t-T}^{t} \lambda^2\,d\lambda = \frac{t^2 - (t-I)^2}{2I}$

$$y(t) = \begin{cases} \frac{t^2}{2I} & 0<t\leq I \\ \frac{t^2-(t-I)^2}{2I} & t\geq I \\ 0 & \text{else} \end{cases}$$

#4

$$x(t) \longrightarrow \boxed{h_1(t)} \xrightarrow{v(t)} \boxed{h_2(t)} \longrightarrow y(t)$$

<u>Note</u> $v(t) = h_1(t) * x(t)$

$$y(t) = h_2(t) * v(t) = h_2(t) * \left(h_1(t) * x(t)\right) = \left(h_1(t) * h_2(t)\right) * x(t)$$

so $h(t) = h_1(t) * h_2(t)$

a) $h_1(t) = \delta(t)$ $\quad h_2(t) = 2e^{-t}u(t)$

$$h(t) = \int_{-\infty}^{\infty} h_1(t-\lambda)\, h_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} \delta(t-\lambda)\, 2e^{-\lambda} u(\lambda)\, d\lambda$$

$$= \boxed{2e^{-t}u(t) = h(t)}$$

note $h(t) * \delta(t) = h(t)$ (no need to do the math!)

b) $h_1(t) = e^{-t}u(t)$ $\quad h_2(t) = 2\delta(t-1)$

$$h(t) = h_1(t) * h_2(t) = \boxed{2e^{-(t-1)}u(t-1) = h(t)}$$

using LTI property

c) $h_1(t) = e^{-t}u(t)$ $\quad h_2(t) = e^{-t}u(t)$

$$h(t) = \int_{-\infty}^{\infty} h_1(t-\lambda)\, h_2(\lambda)\, d\lambda = \int_{-\infty}^{\infty} e^{-(t-\lambda)} u(t-\lambda)\, e^{-\lambda} u(\lambda)\, d\lambda$$

$$= e^{-t} \int_{-\infty}^{\infty} u(t-\lambda)\, u(\lambda)\, d\lambda = e^{-t} \int_0^t d\lambda = \boxed{te^{-t}u(t) = h(t)}$$

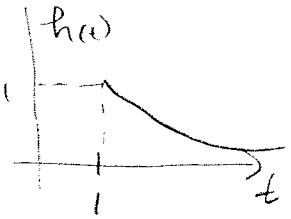
d) $h_1(t) = 2\delta(t-1)$ $\quad h_2(t) = 3\delta(t-2)$

$$h(t) = h_1(t) * h_2(t) = \boxed{6\,\delta(t-3) = h(t)}$$ using LTI

e) $h_1(t) = 2\delta(t-1)$ $\quad h_2(t) = u(t)$

$$h(t) = h_1(t) * h_2(t) = \boxed{2u(t-1) = h(t)}$$ using LTI

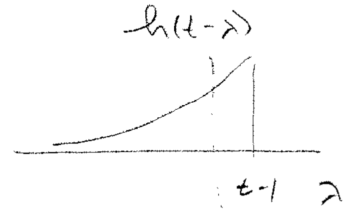
#5 $h(t) = e^{-(t-1)}u(t-1) \quad x(t) = u(t) - u(t-1) + u(t-3)$

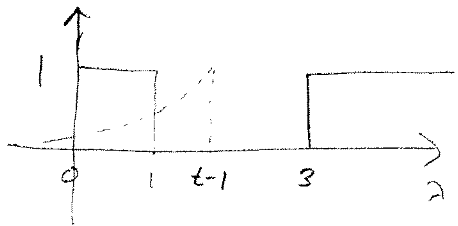


$h(1) = h(t-\lambda)$

$1 = t - \lambda$

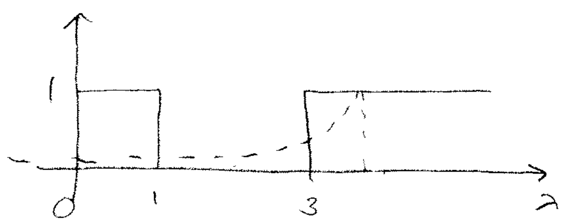
$\lambda = t-1$





$2 \le t \le 4$

$$y(t) = \int_0^1 e^{-(t-\lambda-1)}\,d\lambda = e^{-(t-1)} \int_0^1 e^{\lambda}\,d\lambda = e^{-(t-1)}\left[e^1 - 1\right]$$



$t \ge 4$

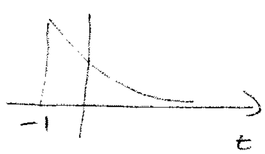
$$y(t) = \int_0^1 e^{-(t-\lambda-1)}\,d\lambda + \int_3^{t-1} e^{-(t-\lambda-1)}\,d\lambda$$

$$= e^{-(t-1)}\left[e^1 - 1\right] + e^{-(t-1)}\left[e^{\lambda}\right]_3^{t-1}$$

$$= e^{-(t-1)}\left[e^1 - 1\right] + e^{-(t-1)}\left[e^{t-1} - e^3\right]$$

$$\boxed{y(t) = \begin{cases} e^{-(t-1)}\left[e^1 - 1\right] & 2 \le t \le 4 \\ e^{-(t-1)}\left[e^1 - 1\right] + e^{-(t-1)}\left[e^{t-1} - e^3\right] & 4 \le t \end{cases}}$$

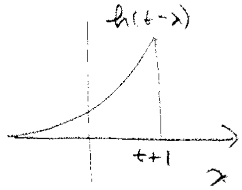
#6 $h(t) = e^{-(t+1)}u(t+1)$ $\quad x(t) = 2[u(t) - u(t-1)] + 3[u(t-3) - u(t-4)]$

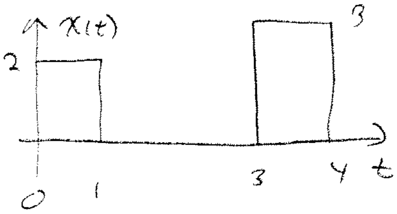


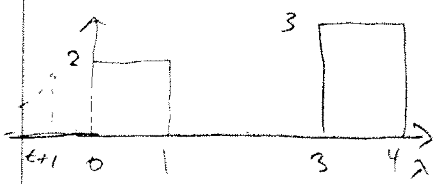
$h(-1) = h(t-\lambda)$

$-1 = t - \lambda$

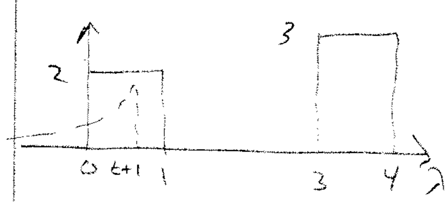
$\lambda = t+1$



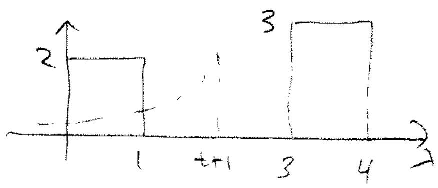




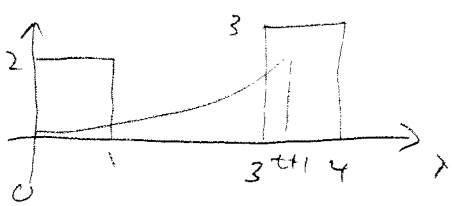
$t \le -1 \qquad y(t) = 0$



$-1 \le t \le 0 \qquad y(t) = \int_0^{t+1} e^{-(t-\lambda+1)}\, 2\, d\lambda$

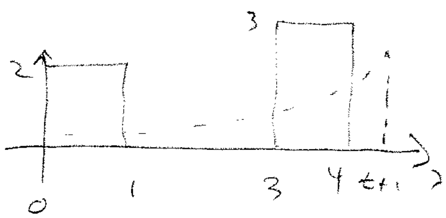


$0 \le t \le 2 \qquad y(t) = \int_0^1 e^{-(t-\lambda+1)}\, 2\, d\lambda$



$2 \le t \le 3$

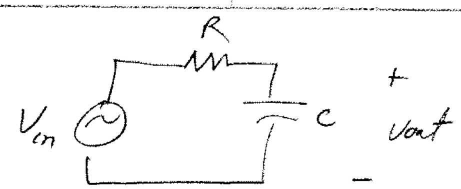
$$y(t) = \int_0^1 e^{-(t-\lambda+1)}\, 2\, d\lambda + \int_3^{t+1} e^{-(t-\lambda+1)}\, 3\, d\lambda$$



$t \ge 3$

$$y(t) = \int_0^1 e^{-(t-\lambda+1)}\, 2\, d\lambda + \int_3^4 e^{-(t-\lambda+1)}\, 3\, d\lambda$$

#7



(a)

$$\frac{V_{in} - V_{out}}{R} = C\frac{dV_{out}}{dt} \qquad CR\frac{dV_{out}}{dt} + V_{out} = V_{in}$$

$$\frac{dV_{out}}{dt} + \frac{1}{RC}V_{out} = \frac{1}{RC}V_{in} \qquad \frac{d}{dt}\left(e^{t/RC}V_{out}\right) = \frac{1}{RC}V_{in}\, e^{t/RC}$$

$$\int_{-\infty}^{t}\frac{d}{dt}\left(e^{t/RC}V_{out}(t)\right)dt = e^{t/RC}V_{out} = \int_{-\infty}^{t}\frac{1}{RC}e^{\lambda/RC}V_{in}(\lambda)\,d\lambda$$

for impulse response $V_{in}(t) = \delta(t)$ $V_{out}(t) = h(t)$

$$e^{+t/RC}h(t) = \int_{-\infty}^{t}\frac{1}{RC}e^{\lambda/RC}\delta(\lambda)\,d\lambda = \frac{1}{RC}\,u(t)$$

$$\boxed{h(t) = \frac{1}{RC}e^{-t/RC}u(t)} \qquad \boxed{\tau = RC}$$

or $\dfrac{V_{out}(s)}{V_{in}(s)} = \dfrac{1/Cs}{R + 1/Cs} = \dfrac{1}{RCs+1} = \dfrac{1}{RC}\,\dfrac{1}{s+1/RC} \Rightarrow h(t) = \dfrac{1}{RC}e^{-t/RC}u(t)$

(b) $h(t) = \frac{1}{\tau}e^{-t/\tau}u(t)$

$$y(t) = h(t) * u(t) = \int_{-\infty}^{\infty}h(\lambda)u(t-\lambda) = \int_0^t \frac{1}{\tau}e^{-t/\tau}d\tau = -e^{-t/\tau}\Big|_0^t$$

$$\boxed{y(t) = \left[1 - e^{-t/\tau}\right]u(t)}$$

For rise time $0.9 = 1 - e^{-t_9/\tau}$ or $0.1 = e^{-t_9/\tau}$

$0.1 = 1 - e^{-t_1/\tau}$ or $0.9 = e^{-t_1/\tau}$

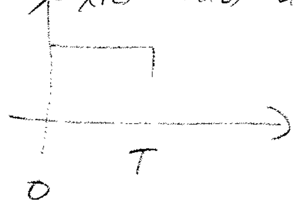
taking ratios $9 = e^{-(t_1 - t_9)/\tau} = e^{(t_9 - t_1)/\tau} = e^{t_r/\tau}$

$\ln 9 = t_r/\tau$ $\boxed{t_r = \tau \ln 9}$

#7 continued)

(c) up to you, but c should be small

(d) $x(t) = u(t) - u(t-T)$



for $x(t) = u(t)$ $\quad y(t) = \left[1 - e^{-t/\tau}\right] u(t)$

since LTI for $x(t) = u(t) - u(t-T)$, $\boxed{y(t) = \left[1 - e^{-t/\tau}\right] u(t) - \left[1 - e^{-(t-T)/\tau}\right] u(t-T)}$

(e) see plots

(F) For $x(t) = A\left[u(t) - u(t-T)\right]$ the output is

$$y(t) = A\left[1 - e^{-t/\tau}\right] u(t) - A\left[1 - e^{-(t-T)/\tau}\right] u(t-T)$$

$$y(T) = A\left[1 - e^{-T/\tau}\right]$$

For $T/\tau \ll 1$ $\quad e^{-T/\tau} \approx 1 - T/\tau$

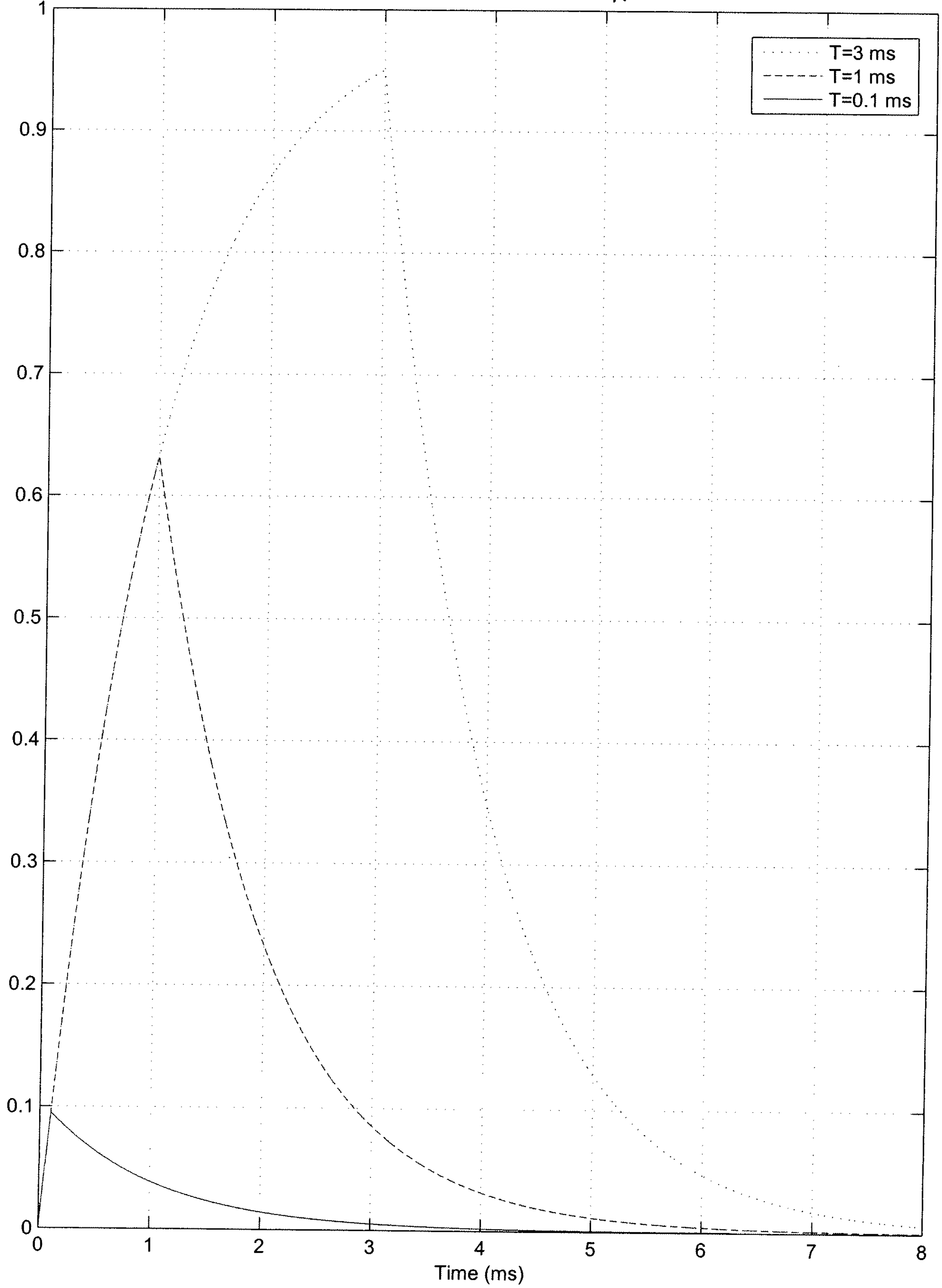
$$y(T) \approx A\left[1 - (1 - T/\tau)\right] = \boxed{\frac{AT}{\tau} = y(T)}$$

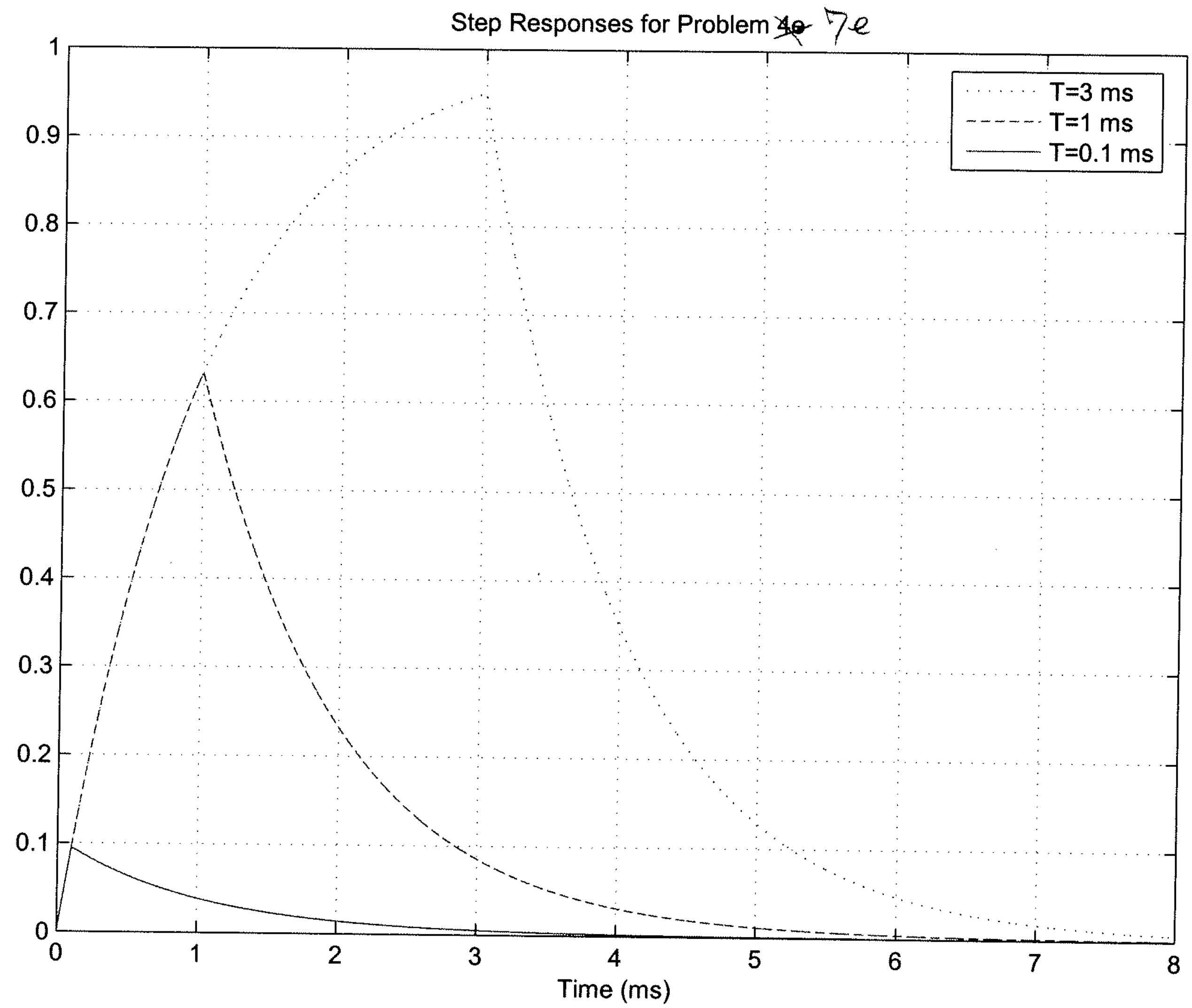
```
%
%  step response plot for homework 3 (prelab for impulse response lab)
%
t = linspace(0,0.008,10000);
tau = 0.001;
%
  T = 0.003;
  y1 = (1-exp(-t/tau)).*unit_step(t,0)-(1-exp(-(t-T)/tau)).*unit_step(t,T);
  T = 0.001;
  y2 = (1-exp(-t/tau)).*unit_step(t,0)-(1-exp(-(t-T)/tau)).*unit_step(t,T);
  T = 0.0001;
  y3 = (1-exp(-t/tau)).*unit_step(t,0)-(1-exp(-(t-T)/tau)).*unit_step(t,T);
  orient tall

  plot(t*1000,y1,':',t*1000,y2,'--',t*1000,y3,'-'); grid;
  legend('T=3 ms','T=1 ms','T=0.1 ms');
  xlabel('Time (ms)'); title('Step Responses for Problem 8e');

  figure;
  orient tall
  subplot(3,1,1); plot(t*1000,y1); grid; ylabel('T = 3 ms'); xlabel('Time (ms)');
  title('Results for Problem 8e');
  subplot(3,1,2); plot(t*1000,y2); grid; ylabel('T = 1 ms'); xlabel('Time (ms)');
  subplot(3,1,3); plot(t*1000,y3); grid; ylabel('T = 0.1 ms'); xlabel('Time (ms)');
```

Step Responses for Problem ~~8~~7e





Step Responses for Problem ~~4e~~ 7e
T=3 ms
T=1 ms
T=0.1 ms
Time (ms)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
1
2
3
4
5
6
7
8

Results for Problem ~~8~~7e