# ECE 300
## Signals and Systems
Homework 7

**Due Date:** Tuesday April 24 *at the beginning of class*

**Exam 2, Thursday April 26**

**Problems:**

1. A periodic signal $x(t)$ is the input to an LTI system with output $y(t)$. The signal $x(t)$ has period 2 seconds, and is given over one period as

$$x(t) = e^{-t} \quad 0 < t < 2$$

$x(t)$ has the Fourier series representation

$$x(t) = \sum_k \frac{0.4323}{1 + jk\pi} e^{jk\pi t}$$

The system is an ideal lowpass filter that eliminates all signals with frequency content higher than 1.25 Hz.

a) Find the average power in $x(t)$.

b) Determine an expression for the output, $y(t)$. Your expression for $y(t)$ must be real.

c) Determine the average power in $y(t)$.

d) Plot the spectrum (magnitude and phase) for $x(t)$. Include the DC through second harmonic. Accurately label your plot.

2. Assume $x(t) = t^2 \quad -\pi \le t \le \pi$ with Fourier Series representation

$$x(t) = \sum_k a_k e^{jkt}$$

where

$$a_k = \begin{cases} \dfrac{\pi^2}{3} & k = 0 \\ \dfrac{2(-1)^k}{k^2} & k \neq 0 \end{cases}$$

a) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies outside the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$? (Note: your answers must be real, no $e^{j\alpha}$ terms.)

b) Assume $x(t)$ is the input to a system that eliminates all signals with frequencies in the range 0.5 to 0.7 Hz. What is the output of the system $y(t)$ and what fraction of the average power in $x(t)$ is in $y(t)$?

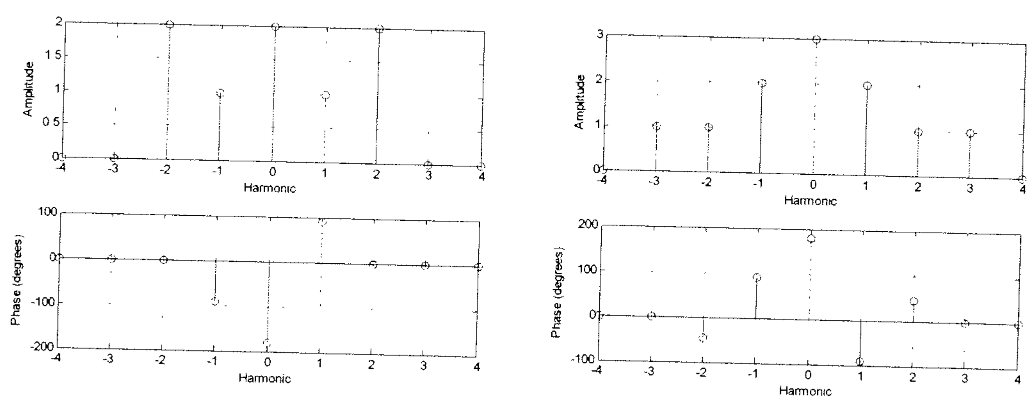
3. K & H, Problem 5.1. Use the example we did in class to get the Fourier series coefficients for part c.

4. K & H, Problem 5.3.

5. K & H, Problem 5.12. Note that $y(t) = x(t) - x(t-1)$. You need to write $c_k^y$ in terms of $c_k^x$.

6. K & H, Problme 5.13.

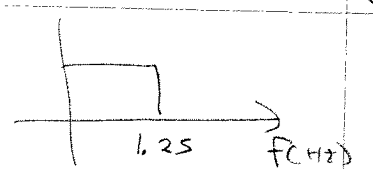
7. The output of a LTI system, $y(t)$, has the following spectrum shown on the left, while the system transfer function, $H(k\omega_o)$, has the spectrum shown on the right. Assume all angles are multiples of 45 degrees.




a) Determine (sketch) the spectrum (magnitude and phase) of the input to the system, $x(t)$.

b) If $x(t)$ has the fundamental period $T = 2$ seconds, determine an analytical expression for $x(t)$ in terms of sine, cosines, and constants.

**#1** $x(t) = e^{-t} \quad 0 < t < 2 \qquad x(t) = \sum_k \frac{0.4323}{1+jk\pi} e^{jk\pi t}$




(a) $P_{ave} = \frac{1}{2}\int_0^2 e^{-2t}\,dt$

$= \frac{1}{2} \frac{e^{-2t}}{-2}\Big|_0^2 = \frac{1}{4}(1-e^{-4}) = \boxed{0.24542 = P_{ave}}$

(b)

| K | $kf_0$ |
|---|---|
| 0 | 0 |
| 1 | ½ |
| 2 | 1 |
| 3 | 3/2 |
| 4 | 2 |

$c_0 = 0.4323$

$c_1 = \frac{0.4323}{1+j\pi} = 0.1311 \angle -72.34°$

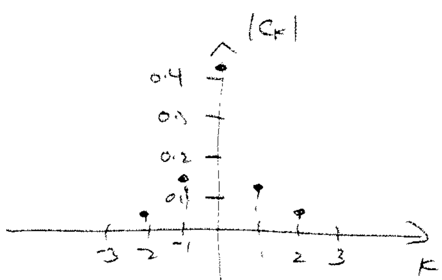
$c_2 = \frac{0.4323}{1+j2\pi} = 0.0680 \angle -80.96°$

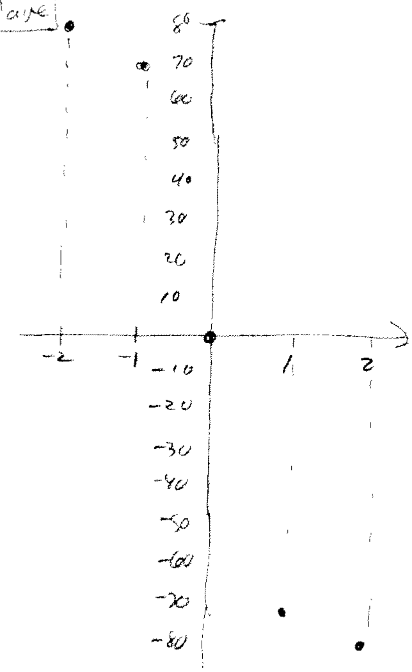
$y(t) = c_0 + 2|c_1|\cos(\omega_0 t + \angle c_1) + 2|c_2|\cos(2\omega_0 t + \angle c_2)$

$\boxed{y(t) = 0.4323 + 0.2622\cos(\pi t - 72.34°) + 0.1360\cos(2\pi t - 80.96°)}$

(c) $P^y_{ave} = |c_0|^2 + 2|c_1|^2 + 2|c_2|^2 = \boxed{0.2305 = P^y_{ave}}$

(d)

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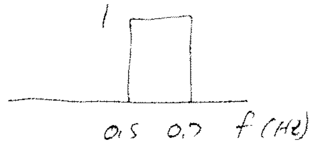
#2 $x(t) = t^2 \quad -\pi \le t \le \pi \qquad x(t) = \frac{\pi^2}{3} + \sum_{k \neq 0} \frac{2(-1)^k}{k^2} e^{jkt}$

a)




Remove all signals with frequencys outside 0.5 to 0.7 Hz

$f_0 = \frac{1}{T_0} = \frac{1}{2\pi} = 0.15915$

| $K$ | $Kf_0$ |
|---|---|
| 0 | 0 |
| 1 | 0.15915 |
| 2 | 0.31831 |
| 3 | 0.47746 |
| 4 | 0.63662 |
| 5 | 0.79577 |

$c_4 = \frac{2(-1)^4}{4^2} = 0.125 \angle 0°$

$y(t) = 2|c_4|\cos(4\omega_0 t + \angle c_4) = 2(0.125)\cos(4t) = \boxed{0.25\cos(4t) = y(t)}$

$P^x_{ave} = \frac{1}{2\pi}\int_{-\pi}^{\pi} t^4 dt = \frac{1}{2\pi}\frac{t^5}{5}\Big|_{-\pi}^{\pi} = \frac{1}{10\pi} 2\pi^5 = \frac{\pi^4}{5} = \boxed{19.4818 = P^x_{ave}}$

$P^y_{ave} = 2|c_4|^2 = 0.03125 \qquad \frac{P^x_{ave}}{P^y_{ave}} = \frac{0.03125}{19.48} = 0.0016 = \boxed{0.16\% = \frac{P^x_{ave}}{P^y_{ave}}}$

b) $y(t) = t^2 - 0.25\cos(4t)$

$P^y_{ave} = P^x_{ave} - 0.03125 = 19.45057 \qquad \frac{P^y_{ave}}{P^x_{ave}} = \frac{19.45057}{19.4818} = 0.9984$

$= \boxed{99.84\% = \frac{P^x_{ave}}{P^y_{ave}}}$

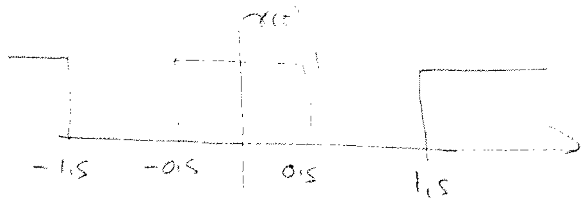
#3 (5.1)

$$H(\omega) = \begin{cases} 1 & 2 \le |\omega| \le 7 \\ 0 & \text{all other } \omega \end{cases} \qquad x(t) \rightarrow \boxed{H} \rightarrow y(t)$$

a) $x(t) = 2 + 3\cos(3t) - 5\sin(6t - 30^\circ) + 4\cos(13t - 20^\circ)$

b) $x(t) = 1 + \sum_{K=1}^{\infty} \frac{1}{K}\cos(2Kt)$

c) $x(t)$ shown below




a) $y(t) = 2H(0) + 3|H(3)|\cos(3t + \measuredangle H(3)) - 5|H(6)|\sin(6t - 30^\circ + \measuredangle H(6))$
$+ 4|H(13)|\cos(13t - 20^\circ - \measuredangle H(13))$

$$\boxed{y(t) = 3\cos(3t) - 5\sin(6t - 30^\circ)}$$

b) $y(t) = 1H(0) + \sum_{K=1}^{\infty} \frac{1}{K}|H(2K)|\cos(2Kt + \measuredangle H(2K))$

$$\boxed{y(t) = \cos(2t) + \frac{1}{2}\cos(4t) + \frac{1}{3}\cos(6t)}$$

c) $C_K^x = \frac{A\tau}{T_0}\operatorname{sinc}\left(K\frac{\tau}{T_0}\right) = \frac{(1)(1)}{2}\operatorname{sinc}\left(K\frac{1}{2}\right) = \frac{1}{2}\operatorname{sinc}\left(\frac{K}{2}\right)$

$T_0 = 2 \qquad \omega_0 = \frac{2\pi}{T_0} = \pi$

$x(t) = \sum_K \frac{1}{2}\operatorname{sinc}\left(\frac{K}{2}\right)e^{jK\pi t}$

For $K=0$ $\omega=0$
$K=1$ $\omega=\pi$
$K=2$ $\omega=2\pi$
$K=3$ $\omega=3\pi$
} only terms

$C_1^y = C_1^x H(\pi) = \frac{1}{2}\operatorname{sinc}\left(\frac{1}{2}\right) = \frac{1}{\pi}$

$C_2^y = C_2^x H(2\pi) = \frac{1}{2}\operatorname{sinc}\left(\frac{2}{2}\right) = 0$

$y(t) = 2|C_1^y|\cos(1\omega_0 t + \measuredangle C_1^y) = \boxed{\frac{2}{\pi}\cos(\pi t) = y(t)}$

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#4

5.3 $H(\omega) = \frac{1}{j\omega + 1}$ $\quad x \to \boxed{H} \to y$

a) $x(t) = \cos(t)$

b) $x(t) = \cos(t + 45^\circ)$

$\omega_0 = 1 \qquad H(j\omega_0) = \frac{1}{1+j} = \frac{1}{\sqrt{2}} \measuredangle -45^\circ$

a) $y(t) = |H(j\omega_0)| \cos(\omega_0 t + \measuredangle H(j\omega_0)) = \boxed{\frac{1}{\sqrt{2}} \cos(t - 45^\circ) = y(t)}$

b) $y(t) = |H(j\omega_0)| \cos(\omega_0 t + \measuredangle H(j\omega_0) + 45^\circ) = \boxed{\frac{1}{\sqrt{2}} \cos(t) = y(t)}$

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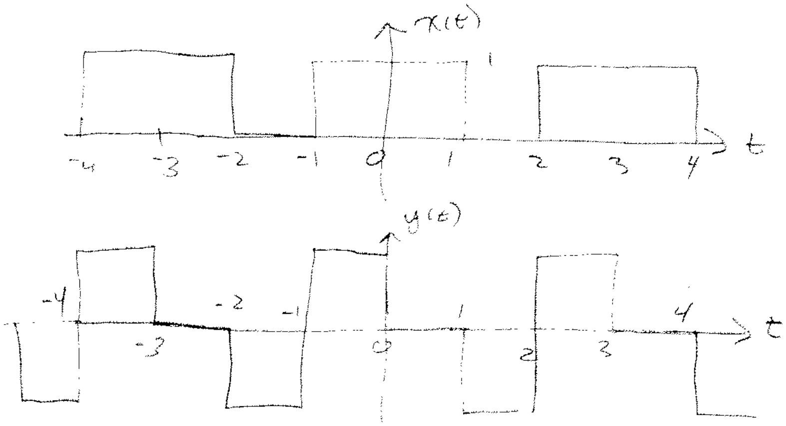
#5

(5.12) $H(\omega) = b - a e^{j\omega c} \qquad -\infty < \omega < \infty$

a, b, c are real numbers $\qquad x(t) \longrightarrow \boxed{H} \longrightarrow y(t)$




$y(t) = x(t) - x(t-1)$

$$C_k^x = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt$$

$$C_k^y = \frac{1}{T_0}\int_{T_0} y(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0}\int_{T_0} \left[x(t) - x(t-1)\right] e^{-jk\omega_0 t}\, dt$$

$$= \underbrace{\frac{1}{T_0}\int_{T_0} x(t)\, e^{-jk\omega_0 t}\, dt}_{C_k^x} - \frac{1}{T_0}\int_{T_0} x(t-1)\, e^{-jk\omega_0 t}\, dt$$

let $\lambda = t-1 \qquad \lambda + 1 = t$

$d\lambda = dt$

limits don't change since still integrating over one period

$$-\frac{1}{T_0}\int_{T_0} x(\lambda)\, e^{-jk\omega_0(\lambda+1)}\, d\lambda$$

$$= -e^{-jk\omega_0}\left[\frac{1}{T_0}\int_{T_0} x(\lambda)\, e^{-jk\omega_0 \lambda}\, d\lambda\right]$$

$$= -e^{-jk\omega_0}\, C_k^x$$

So $C_k^y = C_k^x - e^{-jk\omega_0} C_k^x = \left(1 - e^{-jk\omega_0}\right) C_k^x$

but $C_k^y = C_k^x H(k\omega_0) \Rightarrow H(k\omega_0) = 1 - e^{-jk\omega_0}$

$\Rightarrow \boxed{a = b = 1 \quad c = -1}$

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#6 5.13 $x(t) = 1 + 4\cos(2\pi t) + 8\sin(3\pi t - 90°)$

$y(t) = 2 - 2\sin(2\pi t)$ $\qquad x \rightarrow \boxed{H} \rightarrow y$

what is H?

(b) $y(t) = 2 - 2\sin(2\pi t) = 1\,H(0) + 4\,|H(2\pi)|\cos(2\pi t + \measuredangle H(2\pi)) + 8\,|H(3\pi)|\sin(3\pi t - 90° + \measuredangle H(3\pi))$

$H(0) = 2 \qquad H(3\pi) = 0$

$|H(2\pi)| = \frac{1}{2} \qquad \measuredangle H(2\pi) = +90°$

$$\boxed{H(0) = 2 \qquad H(2\pi) = \tfrac{1}{2}e^{j\pi/2} \qquad H(3\pi) = 0}$$

(a) we can only determine $H(\omega)$ at $\omega = 0,\ \omega = 2\pi,\ \omega = 3\pi$)

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#7 (continued)

$|c_2^x| = \frac{2}{1}$ $\quad \angle c_2^x = (0) - (90^\circ) = -45^\circ$

$|c_3^x| = \frac{0}{1} = 0$ $\quad \angle c_3^x = (0) - (0) = 0^\circ$

so $c_0^x = \frac{2}{3} \angle 0^\circ$

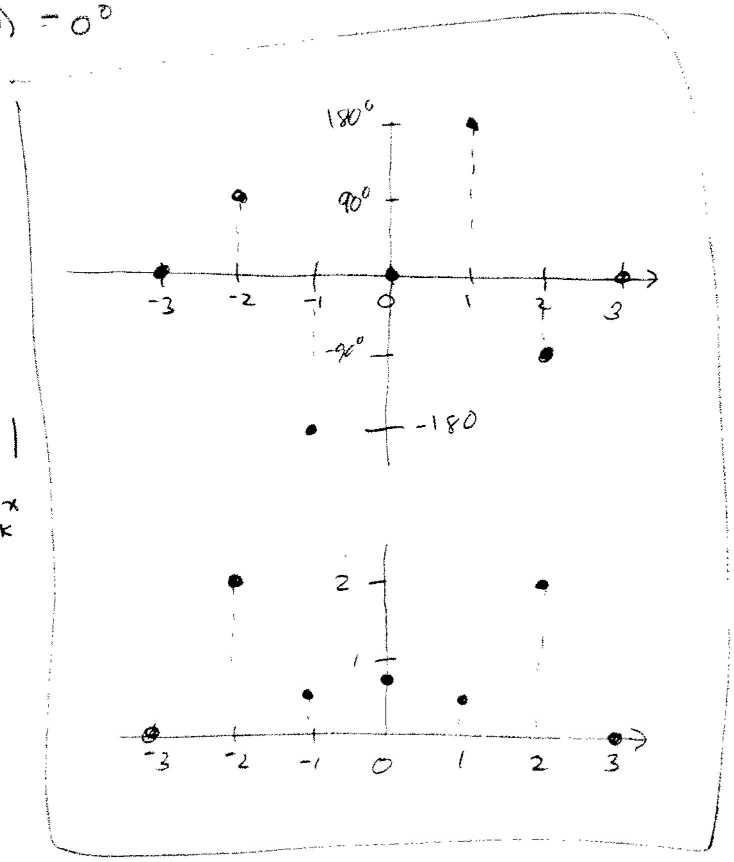
$c_1^x = \frac{1}{2} \angle 180^\circ$

$c_2^x = 2 \angle -90^\circ$

$c_k^x = 0 \quad k \geq 3$

use the relationships $|c_k^x| = |c_{-k}^x|$

$\angle c_{-k}^x = -\angle c_k^x$




$x(t) = c_0^x + 2|c_1^x| \cos(\omega_0 t + \angle c_1^x) + 2|c_2^x| \cos(2\omega_0 t + \angle c_2^x)$

$\omega_0 = \frac{2\pi}{2} = \pi$

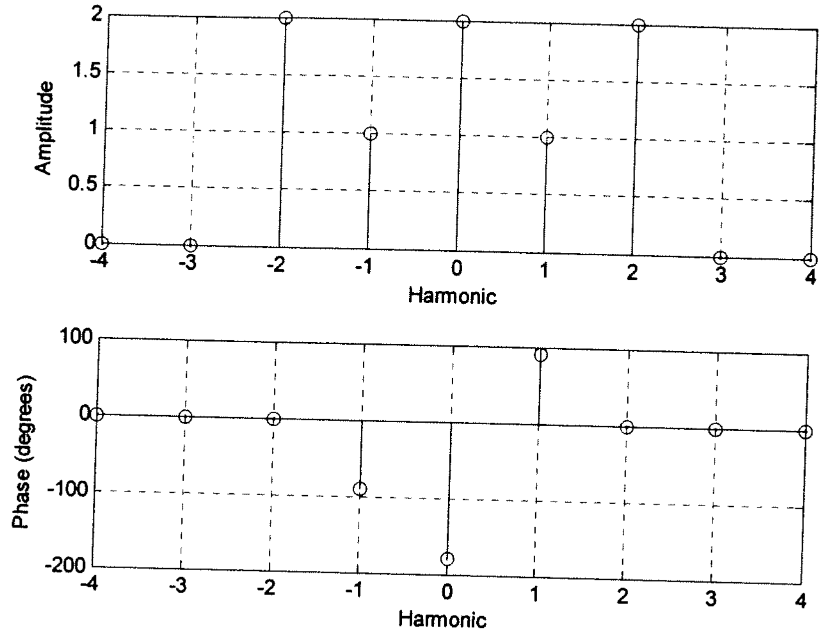
$x(t) = \frac{2}{3} + \cos(\pi t + 180^\circ) + 4\cos(2\pi t - 45^\circ)$

$x(t) = \frac{2}{3} - \cos(\pi t) + 4\cos(2\pi t - 45^\circ)$

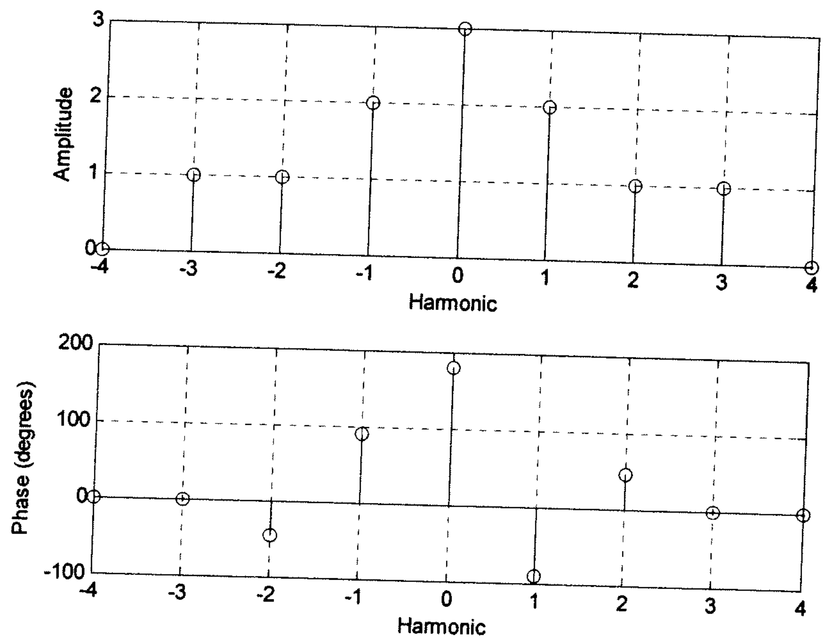
Problem set #7 ECE-300

#7

8. The output of a LTI system, $y(t)$, has the following spectrum:




The system transfer function, $H(k\omega_o)$, has the following spectrum:




$$c_k^y = c_k^x H(jk\omega_o) \quad \text{or} \quad c_k^x = \frac{c_k^y}{H(jk\omega_o)} = \frac{|c_k^y| \angle c_k^y}{|H(jk\omega_o)| \angle H(jk\omega_o)}$$

$$|c_k^x| = \frac{|c_k^y|}{|H(jk\omega_o)|} \qquad \angle c_k^x = \angle c_k^y - \angle H(jk\omega_o)$$

$$|c_0^x| = \frac{2}{3} \qquad \angle c_0^x = (-180^\circ) - (180^\circ) = -360^\circ = 0^\circ$$

$$|c_1^x| = \frac{1}{2} \qquad \angle c_1^x = (90^\circ) - (-90^\circ) = 180^\circ$$

(continued)

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