ECE 300
Signals and Systems
Homework 5

Due Date: *Thursday October 8 at the beginning of class*

1. For the following system models, determine if the model represents a BIBO stable system. If the system is not BIBO stable, give an input $x(t)$ that demonstrates this.

a) $y(t) = \int_{-\infty}^{t} (x(\lambda) - 5) d\lambda$ b) $y(t) = \cos\left(\frac{1}{x(t)}\right)$

c) $y(t) = e^{-|x(t)|}$ d) $y(t) = x(t) + y(t)x(t)$

2. For LTI systems with the following impulse responses, determine if the system is BIBO stable.

a) $h(t) = e^{-t}u(t)$ b) $h(t) = u(t)$ c) $h(t) = u(t) - u(t-10)$ d) $h(t) = \delta(t-1)$

e) $h(t) = \sin(t)u(t)$ f) $h(t) = e^{-t^2}u(t)$ (hint: use your answer to **a)**

3. In this problem we will determine the trigonometric Fourier Series for a full wave rectified signal.

a. Using Euler's identity, show that $\sin(\alpha)\cos(\beta) = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\beta-\alpha)$ and $\sin(\alpha)\sin(\beta) = \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$

b. Show that for $x(t) = V_m \sin\left(\frac{\omega_0}{2}t\right)\ 0 \le t \le T_0$, the trigonometric Fourier series coefficients are given by

$$a_0 = \frac{2V_m}{\pi}$$

$$a_k = \frac{V_m}{\pi}\frac{1}{0.25-k^2}$$

$$b_k = 0$$

4. (**Matlab/Prelab Problem**) Read the **Appendix** and then do the following:

a) Copy the file **Trigonometric_Fourier_Series.m** (from last week's homework) to file **Complex_Fourier_Series.m**.

b) Modify **Complex_Fourier_Series.m** so it computes the average value c_o

c) Modify **Complex_Fourier_Series.m** so it ***directly*** computes c_k for $k=1$ to $k=N$. You are **not** to use the trigonometric Fourier series coefficients for this.

d) Modify **Complex_Fourier_Series.m** so it also computes the Fourier series estimate using the formula

$$x(t) \approx c_o + \sum_{k=1}^{N} 2\,|c_k|\cos(k\omega_o t + \measuredangle c_k)$$

You will probably need to use the Matlab functions **abs** and **angle** for this.

e) Using the code you wrote in part **d,** find the complex Fourier series representation for the following functions (defined over a single period)

$$f_1(t) = e^{-t}u(t) \quad 0 \le t < 3$$

$$f_2(t) = \begin{cases} t & 0 \le t < 2 \\ 3 & 2 \le t < 3 \\ 0 & 3 \le t < 4 \end{cases}$$

$$f_3(t) = \begin{cases} 0 & -2 \le t < -1 \\ 1 & -1 \le t < 2 \\ 3 & 2 \le t < 3 \\ 0 & 3 \le t < 4 \end{cases}$$

Turn in your code. *Be sure to modify your program so any unnecessary code is eliminated (not just commented out). Note that the values of* ***low*** *and* ***high*** *will be different for each of these functions!*

Appendix

In the majority of this course we will be using the complex (or exponential) form of the Fourier series, since it is really easier to do various mathematical things with it once you get used to it.

Exponential Fourier Series If $x(t)$ is a periodic function with fundamental period T, then we can represent $x(t)$ as a Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t}$$

where $\omega_o = \frac{2\pi}{T}$ is the fundamental period, c_o is the average (or DC, i.e. zero frequency) value, and

$$c_o = \frac{1}{T}\int_0^T x(t)dt$$

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}dt$$

If $x(t)$ is a real function, then we have the relationships $|c_k| = |c_{-k}|$ (the magnitude is even) and $\measuredangle c_{-k} = -\measuredangle c_k$ (the phase is odd). Using these relationships we can then write

$$x(t) = c_o + \sum_{k=1}^{\infty} 2|c_k|\cos(k\omega_o t + \measuredangle c_k)$$

This is usually a much easier form to deal with, since it lends itself easily to thinking of a phasor representation of $x(t)$. This will be particularly useful when we starting filtering periodic signals

(#1)

(a) $y(t) = \int_{-\infty}^{t} (x(\lambda) - 5)\, d\lambda$

for $x(t) = 10u(t)$ (bounded)

$y(t) = \int_0^t 5\, d\lambda = 5t$ not bounded not BIBO stable

(b) $y(t) = \cos\left(\frac{1}{x(t)}\right)$ $|y(t)| \le 1$ for all $x(t)$ BIBO stable

(c) $y(t) = e^{-|x(t)|}$ for $|x(t)| \le N$

$|y(t)| \le e^{-N}$ BIBO stable

(d) $y(t) = x(t) + y(t)\,x(t)$

$y(t)\,(1 - x(t)) = x(t)$

$y(t) = \dfrac{x(t)}{1 - x(t)}$ for $x(t) = 1$ (bounded)

$y(t)$ is not bounded

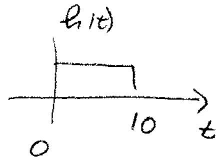
not BIBO stable

#2

a) $h(t) = e^{-t}u(t)$ $\quad \int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} e^{-t}dt = -e^{-t}\Big|_0^{\infty} = 1$ finite, so BIBO stable

b) $h(t) = u(t)$ $\quad \int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} 1\, dt = \infty$ not finite, so not BIBO stable

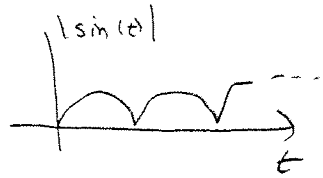
c) $h(t) = u(t) - u(t-10)$



$\int_{-\infty}^{\infty} |h(t)|dt = \int_0^{10} 1\, dt = 10$ finite, so BIBO stable

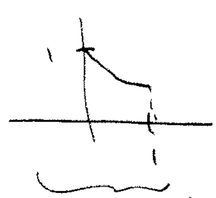
d) $h(t) = \delta(t-1)$ $\quad \int_{-\infty}^{\infty} |h(t)|dt = \int_{-\infty}^{\infty} \delta(t-1)dt = 1$ finite, so BIBO stable

e) $h(t) = \sin(t)\, u(t)$ $\quad \int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} |\sin(t)|dt = \infty$, so not BIBO stable



f) $h(t) = e^{-t^2}$ $\quad$ note $e^{-t^2} \le e^{-t}$ for $t \ge 1$

so $\int_{-\infty}^{\infty} |h(t)|dt = \int_0^{\infty} e^{-t^2}dt \le \underbrace{\int_0^1 e^{-t^2}dt}_{} + \underbrace{\int_1^{\infty} e^{-t}dt}_{\text{finite from part a}}$



clearly finite area

so $\int_{-\infty}^{\infty} |h(t)|dt$ is finite, so BIBO stable

#3 (a) $\sin(\alpha)\cos(\beta) = \left(\frac{e^{j\alpha} - e^{-j\alpha}}{2j}\right)\left(\frac{e^{j\beta} + e^{-j\beta}}{2}\right)$

$$= \frac{1}{2 \cdot 2j}\left[e^{j(\alpha+\beta)} + e^{j(\alpha-\beta)} - e^{-j(\alpha-\beta)} - e^{-j(\alpha+\beta)}\right]$$

$$= \frac{1}{2}\left[\frac{e^{j(\alpha+\beta)} - e^{-j(\alpha+\beta)}}{2j}\right] + \frac{1}{2}\left[\frac{e^{j(\alpha-\beta)} - e^{-j(\alpha-\beta)}}{2j}\right]$$

$$= \frac{1}{2}\sin(\alpha+\beta) + \frac{1}{2}\sin(\alpha-\beta) = \frac{1}{2}\sin(\alpha+\beta) - \frac{1}{2}\sin(\beta-\alpha)$$

$\sin(\alpha)\sin(\beta) = \left(\frac{e^{j\alpha} - e^{-j\alpha}}{2j}\right)\left(\frac{e^{j\beta} - e^{-j\beta}}{2}\right)$

$$= \frac{1}{(2j)(2j)}\left[e^{j(\alpha+\beta)} - e^{j(\alpha-\beta)} - e^{-j(\alpha-\beta)} + e^{-j(\alpha+\beta)}\right]$$

$$= -\frac{1}{2}\left[\frac{e^{j(\alpha+\beta)} + e^{-j(\alpha+\beta)}}{2}\right] + \frac{1}{2}\left[\frac{e^{j(\alpha-\beta)} + e^{-j(\alpha-\beta)}}{2}\right]$$

$$= \frac{1}{2}\cos(\alpha-\beta) - \frac{1}{2}\cos(\alpha+\beta)$$

$x(t)$ is a full wave rectifier $\left\{ \text{or } x(t) = V_m \left|\sin\left(\frac{\omega_0 t}{2}\right)\right|^{0<t\le T} \right.$

$x(t) = V_m \sin\left(\frac{\omega_0}{2} t\right) \quad 0 \le t \le T_0$

$$a_0 = \frac{V_m}{T_0} \int_0^{T_0} \sin\left(\frac{\omega_0}{2} t\right) dt = \frac{V_m}{T_0} \frac{2}{\omega_0} \left[-\cos\left(\frac{\omega_0}{2} t\right) \Big|_0^{T_0} \right]$$

$$= \frac{V_m}{\pi} \left[1 - \cos(\pi) \right] = \boxed{\frac{2V_m}{\pi} = a_0}$$

$$a_K = \frac{2}{T_0} \int_0^{T_0} V_m \sin\left(\frac{\omega_0}{2} t\right) \cos(K\omega_0 t) dt$$

$$= \frac{2V_m}{T_0} \left[\int_0^{T_0} \frac{\sin[(K+\frac{1}{2})\omega_0 t]}{2} dt - \int_0^{T_0} \frac{\sin[(K-\frac{1}{2})\omega_0 t]}{2} dt \right]$$

$$= \frac{V_m}{\pi} \left[\frac{-\cos[(K+\frac{1}{2})\omega_0 t]}{2(K+\frac{1}{2})} \Bigg|_0^{T_0} \right] - \frac{V_m}{\pi} \left[-\frac{\cos[(K-\frac{1}{2})\omega_0 t]}{2(K-\frac{1}{2})} \Bigg|_0^{T_0} \right]$$

$$= \frac{V_m}{\pi} \frac{\left[1 - \cos[(K+\frac{1}{2}) 2\pi]\right]}{2(K+\frac{1}{2})} - \frac{V_m}{\pi} \left[\frac{1 - \cos[(K-\frac{1}{2}) 2\pi]}{2(K-\frac{1}{2})} \right]$$

$$= \frac{V_m}{2\pi} \left[\frac{1 - \cos(2\pi K + \pi)}{(K+\frac{1}{2})} \right] - \frac{V_m}{2\pi} \left[\frac{1 - \cos(2\pi K - \pi)}{K - \frac{1}{2}} \right]$$

$$= \frac{V_m}{2\pi} \left[\frac{2}{K+\frac{1}{2}} \right] - \frac{V_m}{2\pi} \left[\frac{2}{K-\frac{1}{2}} \right] = \frac{2V_m}{2\pi} \left[\frac{(K-\frac{1}{2}) - (K+\frac{1}{2})}{K^2 - \frac{1}{4}} \right] = \boxed{\frac{V_m}{\pi} \frac{1}{\frac{1}{4} - K^2}}$$

$$b_K = \frac{2}{T_0}\int_0^{T_0} V_m \sin\left(\frac{\omega_0}{2} t\right)\sin\left(K\omega_0 t\right) dt$$

$$= \frac{2V_m}{T_0}\int_0^{T_0} \frac{\cos[(K-\tfrac{1}{2})\omega_0 t]}{2(K-\tfrac{1}{2})\omega_0}\, dt - \frac{2V_m}{T_0}\int_0^{T_0} \frac{\cos[(K+\tfrac{1}{2})\omega_0 t]}{2(K+\tfrac{1}{2})\omega_0}\, dt$$

$$= \frac{V_m}{2\pi}\left[\frac{\sin[(K-\tfrac{1}{2})\omega_0 t]}{(K-\tfrac{1}{2})}\Bigg|_0^{T_0}\right] - \frac{V_m}{2\pi}\left[\frac{\sin[(K+\tfrac{1}{2})\omega_0 t]}{(K+\tfrac{1}{2})}\Bigg|_0^{T_0}\right]$$

$$= \frac{V_m}{2\pi}\left[\cancelto{0}{\frac{\sin[(K-\tfrac{1}{2})2\pi]}{(K-\tfrac{1}{2})}}\right] - \frac{V_m}{2\pi}\left[\cancelto{0}{\frac{\sin[(K+\tfrac{1}{2})2\pi]}{(K+\tfrac{1}{2})}}\right] = 0$$