ECE 300
Signals and Systems
Homework 5

Due Date: Tuesday October 9, 2007 at the beginning of class

Problems:

1. Determine the complex Fourier series coefficients for the following periodic signals:

a) $x(t) = -1 + \cos(2t) + 3\cos\left(4t + \frac{\pi}{4}\right)$

b) $x(t) = \cos^2(2t)$ *(Hint: Use Euler's identify)*

c) $x(t) = 2\cos(3t) + 4\sin\left(6t - \frac{\pi}{3}\right)$

2. Find the Fourier series representation for the signal indicated using hand analysis. Clearly indicate the values of ω_0 and the c_k. *Hints:* (1) *Draw the signal, and then use the sifting property to calculate the* c_k. (2) *If you understand how to do this, there is very little work involved.*

$$x(t) = \sum_{p=-\infty}^{\infty} \delta(t - 3p)$$

3. For the periodic square wave $x(t)$ with period $T_o = 0.5$ and

$$x(t)\begin{cases} 1 & 0 \le t < 0.25 \\ -1 & 0.25 \le t < 0.5 \end{cases}$$

show that the Fourier series coefficients are given by

$$c_k = \begin{cases} \dfrac{-2j}{k\pi} & k \quad odd \\ 0 & k \quad even \end{cases}$$

where $x(t) = \sum_k c_k e^{jk4\pi t}$

4. Simplify each of the following into the form $c_k = \alpha(k)e^{-j\beta(k)}\text{sinc}(\lambda k)$

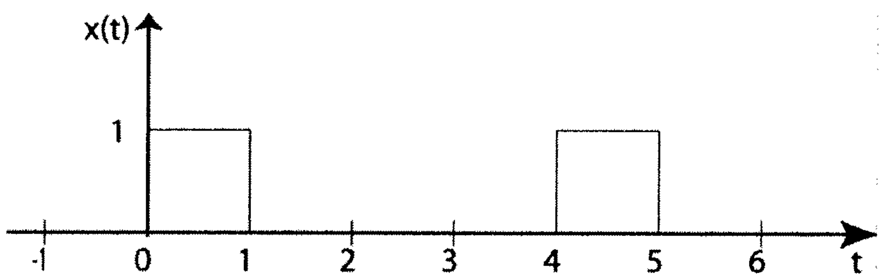
a) $c_k = \dfrac{e^{j7k\pi} - e^{-j2k\pi}}{k\pi j}$

b) $c_k = \dfrac{e^{-j2\pi k} - e^{-j5\pi k}}{jk}$

c) $c_k = \dfrac{e^{j5k} - e^{j2k}}{k}$

Scrambled Answers $c_k = 3\pi e^{-j\frac{7\pi k}{2}}\text{sinc}\left(\dfrac{3k}{2}\right)$, $c_k = 3e^{j(\frac{7}{2}k+\frac{\pi}{2})}\text{sinc}\left(\dfrac{3k}{2\pi}\right)$, $c_k = 9e^{j\frac{5}{2}k\pi}\text{sinc}\left(k\dfrac{9}{2}\right)$

5. For the periodic signal shown below, with period $T = 4$



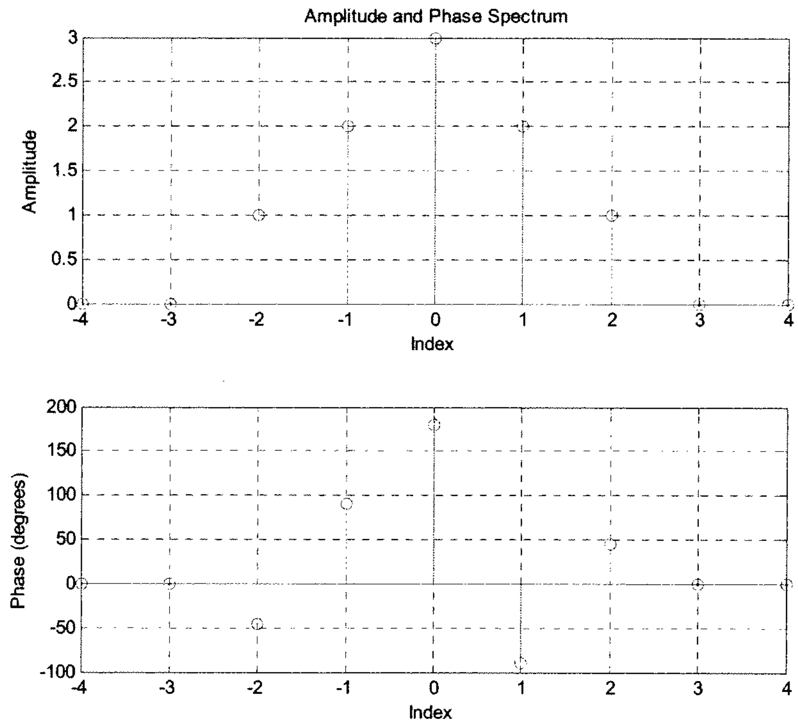
a) Determine the fundamental frequency ω_0.

b) Determine the average value.

c) Determine the average power in the DC component of the signal.

d) Determine an expression for the expansion coefficients, c_k. You must write your expression in terms of the **sinc** function, and possibly a leading phase term.

6. Assume $x(t)$ has the spectrum shown below (all angles are multiples of 45 degrees):



a) What is the ***average value*** of $x(t)$?

b) What is the ***average power*** in $x(t)$?

c) Write an expression for $x(t)$ in terms of cosines (Assume $\omega_0 = 1$). Your expression must be ***real***.

Special Note: We will be using the code you write in the next part for the next few homeworks and labs, so be sure you do this and understand what is going on!

7. (Matlab/PreLab Problem) Read the **Appendix** and then do the following:

a) Copy the file **Trigonometric_Fourier_Series.m** (you wrote this for homework 4) to file **Complex_Fourier_Series.m**.

b) Modify **Complex_Fourier_Series.m** so it computes the average value c_o

c) Modify **Complex_Fourier_Series.m** so it also computes c_k for $k = 1$ to $k = N$

d) Modify **Complex_Fourier_Series.m** so it also computes the Fourier series estimate using the formula

$$x(t) \approx c_o + \sum_{k=1}^{N} 2\,|c_k|\cos(k\omega_o t + \measuredangle c_k)$$

You will probably need to use the Matlab functions **abs** and **angle** for this.

e) Using the code you wrote in part **d,** find the complex Fourier series representation for the following functions (defined over a single period)

$$f_1(t) = e^{-t}u(t) \quad 0 \le t < 3$$

$$f_2(t) = \begin{cases} t & 0 \le t < 2 \\ 3 & 2 \le t < 3 \\ 0 & 3 \le t < 4 \end{cases}$$

$$f_3(t) = \begin{cases} 0 & -2 \le t < -1 \\ 1 & -1 \le t < 2 \\ 3 & 2 \le t < 3 \\ 0 & 3 \le t < 4 \end{cases}$$

These are the same functions you used for the trigonometric Fourier series. Use N = 10 and turn in your plots for each of these functions. Also, turn in your Matlab program for one of these. *Note that the values of **low** and **high** will be different for each of these functions!*

Appendix

In the majority of this course we will be using the complex (or exponential) form of the Fourier series, since it is really easier to do various mathematical things with it once you get used to it.

Exponential Fourier Series If $x(t)$ is a periodic function with fundamental period T, then we can represent $x(t)$ as a Fourier series

$$x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jk\omega_o t} = \sum_{k=-\infty}^{\infty} X_k e^{jk\omega_o t}$$

where $\omega_o = \frac{2\pi}{T}$ is the fundamental period, c_o is the average (or DC, i.e. zero frequency) value, and

$$c_o = \frac{1}{T}\int_0^T x(t)dt$$

$$c_k = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}dt$$

If $x(t)$ is a real function, then we have the relationships $|c_k| = |c_{-k}|$ (the magnitude is even) and $\measuredangle c_{-k} = -\measuredangle c_k$ (the phase is odd). Using these relationships we can then write

$$x(t) = c_o + \sum_{k=1}^{\infty} 2|c_k|\cos(k\omega_o t + \measuredangle c_k)$$

This is usually a much easier form to deal with, since it lends itself easily to thinking of a phasor representation of $x(t)$. This will be particularly useful when we starting filtering periodic signals.

(#1) Find c_k for the following!

a) $x(t) = -1 + \cos(2t) + 3\cos(4t + \frac{\pi}{4})$

b) $x(t) = \cos^2(2t)$

c) $x(t) = 2\cos(3t) + 4\sin(6t - \frac{\pi}{3})$

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a) $x(t) = -1 + \cos(2t) + 3\cos(4t + \pi/4)$

$= -1 + \left(\frac{1}{2}e^{j2t} + \frac{1}{2}e^{-j2t}\right) + 3\left(\frac{1}{2}e^{j(4t+\pi/4)} + \frac{1}{2}e^{-j(4t+\pi/4)}\right)$

$= \frac{3}{2}e^{-j\pi/4}e^{-j4t} + \frac{1}{2}e^{-j2t} - 1 + \frac{1}{2}e^{j2t} + \frac{3}{2}e^{j\pi/4}e^{j4t}$

$$\boxed{\begin{aligned} c_0 &= -1 \\ c_1 &= \tfrac{1}{2} = c_{-1}^* \\ c_2 &= \tfrac{3}{2}e^{j\pi/4} = c_{-2}^* \end{aligned}}$$

assuming $\omega_0 = 2$

b) $x(t) = \cos^2(2t) = \left(\frac{e^{j2t} + e^{-j2t}}{2}\right)^2 = \frac{e^{j4t} + 2 + e^{-j4t}}{4}$

$$\boxed{\begin{aligned} c_0 &= \tfrac{1}{2} \\ c_2 &= \tfrac{1}{4} = c_{-2}^* \end{aligned}}$$

assuming $\omega_0 = 2$

if you assume $\omega_0 = 4$, then $c_0 = 1/2$, $c_1 = 1/4 = c_{-1}^*$

c) $x(t) = 2\cos(3t) + 4\sin(6t - \pi/3)$

$= 2\left(\frac{e^{j3t}}{2} + \frac{e^{-j3t}}{2}\right) + \frac{4}{2j}\left(e^{j(6t-\pi/3)} - e^{-j(6t-\pi/3)}\right)$

$= e^{j3t} + e^{-j3t} + 2e^{-j\pi/2}e^{-j\pi/3}e^{j6t} - 2e^{-j\pi/2}e^{j\pi/3}e^{-j6t}$

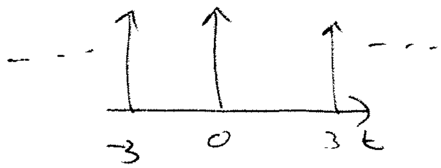
$= e^{j3t} + e^{-j3t} + 2e^{-j\frac{5\pi}{6}}e^{j6t} + 2e^{j\frac{5\pi}{6}}e^{-j6t}$ $\quad -1 = e^{j\pi}$

$$\boxed{\begin{aligned} c_0 &= 0 \\ c_1 &= 1 = c_{-1}^* \\ c_2 &= 2e^{-j5\pi/6} = c_{-2}^* \end{aligned}}$$

assuming $\omega_0 = 3$

#2 $x(t) = \sum_{p=-\infty}^{\infty} \delta(t-3p)$

$T_0 = 3$

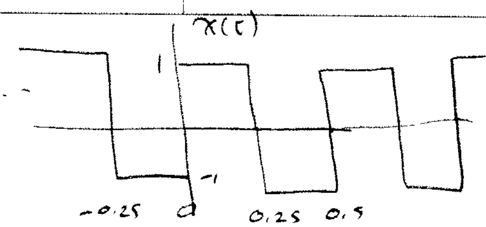


$$C_K = \frac{1}{T_0}\int_{-3/2}^{3/2} \delta(t)\, e^{-jk\omega_0 t}\, dt = \frac{1}{T_0} = \frac{1}{3}$$

$$\boxed{x(t) = \sum_{K} \frac{1}{3} e^{jK\frac{2\pi}{3}t}}$$

#3

$$x(t) = \begin{cases} 1 & 0 \le t \le 0.25 \\ -1 & 0.25 \le t \le 0.5 \end{cases}$$



$T_0 = \frac{1}{2} \quad \omega_0 = \frac{2\pi}{T_0} = 4\pi$

$c_0 = 0$ (average value is zero)

$$c_k = \frac{1}{T_0}\int_{-0.25}^{0} (-1)\, e^{-jk\omega_0 t}\, dt + \frac{1}{T_0}\int_{0}^{0.25} (1)\, e^{-jk\omega_0 t}\, dt$$

$$= 2\int_{-0.25}^{0} (-1)\, e^{-jk4\pi t}\, dt + 2\int_{0}^{0.25} (1)\, e^{-jk4\pi t}\, dt$$

$$= -2 \left. \frac{e^{-jk4\pi t}}{-jk4\pi} \right|_{-0.25}^{0} + 2 \left. \frac{e^{-jk4\pi t}}{-jk4\pi} \right|_{0}^{0.25}$$

$$= \frac{2}{jk4\pi}\left[1 - e^{-jk\pi}\right] - \frac{2}{jk4\pi}\left[e^{+jk\pi} - 1\right]$$

$$= \frac{4}{jk4\pi}\left[1 - e^{jk\pi}\right] = \frac{1}{jk\pi}\left[1 - (-1)^k\right]$$

$$= = \frac{-j}{k\pi}\left[1 - (-1)^k\right]$$

$$\boxed{a_k = \begin{cases} \dfrac{-2j}{k\pi} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}}$$

#4 (a) $c_K = \dfrac{e^{j7K\pi} - e^{-j2K\pi}}{jK\pi} = \dfrac{e^{j\frac{5}{2}\pi K}\left[e^{j\frac{9}{2}\pi K} - e^{-j\frac{9}{2}\pi K}\right]}{jK\pi}$

$= \dfrac{e^{j\frac{5}{2}\pi K}}{K\pi}\, 2 \left[\dfrac{e^{j\frac{9}{2}\pi K} - e^{-j\frac{9}{2}\pi K}}{2j}\right]$

$= \dfrac{2e^{j\frac{5}{2}\pi K}}{K\pi} \sin\left(\frac{9}{2}\pi K\right)$

$= 2e^{j\frac{5}{2}\pi K} \dfrac{\sin\left(\frac{9}{2}\pi K\right)}{\pi K\left(\frac{9}{2}\right)\left(\frac{2}{9}\right)} = \boxed{9e^{j\frac{5}{2}\pi K}\,\mathrm{sinc}\left(K\frac{9}{2}\right) = c_K}$

(b) $c_K = \dfrac{e^{-j2K\pi} - e^{-j5\pi K}}{jK} = \dfrac{e^{-j\frac{7}{2}\pi K}\left[e^{+j\frac{3}{2}\pi K} - e^{-j\frac{3}{2}\pi K}\right]}{jK}$

$= \dfrac{2e^{-j\frac{7}{2}\pi K}}{K} \sin\left(\frac{3}{2}\pi K\right) = 2e^{-j\frac{7}{2}\pi K} \dfrac{\sin\left(\frac{3}{2}\pi K\right)}{K\cdot\left(\frac{3}{2}\pi\right)\left(\frac{2}{3\pi}\right)}$

$= 2e^{-j\frac{7}{2}\pi K}\,\mathrm{sinc}\left(\frac{3K}{2}\right)\cdot\frac{3\pi}{2}$

$\boxed{c_K = 3\pi e^{-j\frac{7}{2}\pi K}\,\mathrm{sinc}\left(\frac{3K}{2}\right)}$

(c) $c_K = \dfrac{e^{j5K} - e^{j2K}}{K} = \dfrac{e^{j\frac{7}{2}K}\left(e^{j\frac{3}{2}K} - e^{-j\frac{3}{2}K}\right)}{K}$

$= \dfrac{2j}{K} e^{j\frac{7}{2}K}\left(\dfrac{e^{j\frac{3}{2}K} - e^{-j\frac{3}{2}K}}{2j}\right) = \dfrac{2j}{K} e^{j\frac{7}{2}K} \sin\left(\frac{3}{2}K\right)$

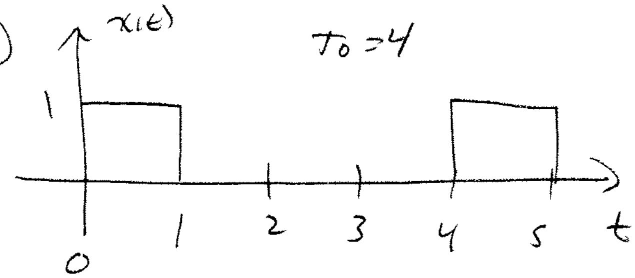
$= \dfrac{2e^{j\left(\frac{7}{2}K + \frac{\pi}{2}\right)}}{K} \sin\left(\frac{3}{2}K\frac{\pi}{\pi}\right) = \dfrac{2}{K} e^{j\left(\frac{7}{2}K+\frac{\pi}{2}\right)} \sin\left(\pi\cdot\frac{3K}{2\pi}\right)$

$= \dfrac{2e^{j\left(\frac{7}{2}K+\frac{\pi}{2}\right)} \sin\left(\pi\cdot\frac{3K}{2\pi}\right)}{K\cdot\frac{\pi}{\pi}\,\frac{3}{3}\,\frac{2\pi}{2\pi}} = \dfrac{2e^{j\left(\frac{7}{2}K+\frac{\pi}{2}\right)} \sin\left(\pi\frac{3K}{2\pi}\right)}{\frac{\pi 3K}{2\pi}\cdot\frac{2\pi}{3\pi}}$

$\boxed{c_K = 3e^{j\left(\frac{7}{2}K+\frac{\pi}{2}\right)}\,\mathrm{sinc}\left(\frac{3K}{2\pi}\right)}$

#5

$x(t)$ $\quad T_0 = 4$



a) $\omega_0 = \frac{2\pi}{T_0} = \frac{2\pi}{4} = \boxed{\frac{\pi}{2} = \omega_0}$

b) $\bar{x} = \frac{1}{T_0}\int_{T_0} x(t)\,dt = \frac{1}{4}\int_0^1 1\,dt = \boxed{\frac{1}{4} = \bar{x}}$

c) $c_0 = \frac{1}{4}$ $\quad \boxed{P_0 = \frac{1}{16} = c_0^2}$

d) $c_K = \frac{1}{T_0}\int_{T_0} x(t)\, e^{-jK\omega_0 t}\,dt = \frac{1}{4}\int_0^1 e^{-jK\frac{\pi}{2}t}\,dt$

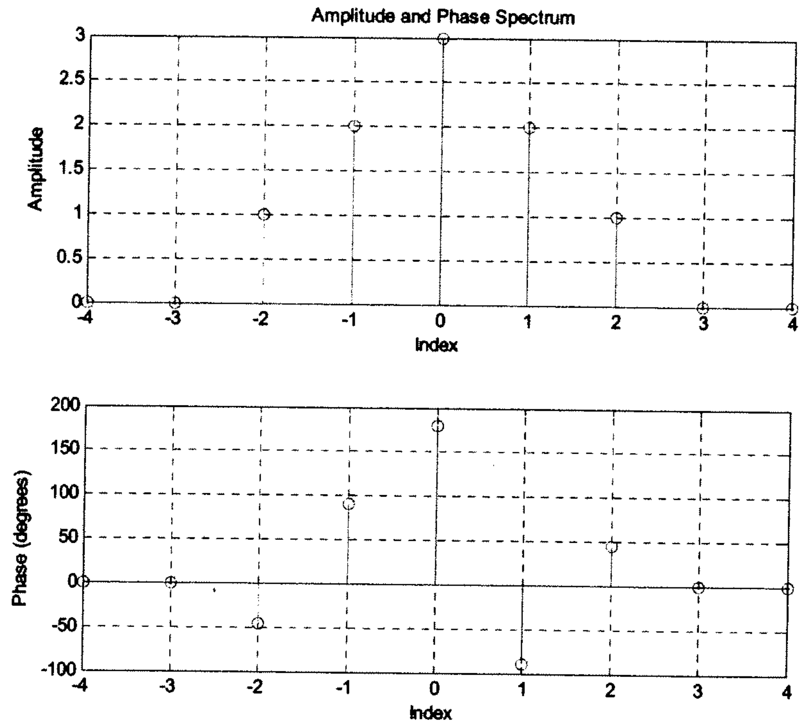
$$= \frac{1}{4}\left.\frac{e^{-jK\frac{\pi}{2}t}}{-jK\frac{\pi}{2}}\right|_0^1 = \frac{e^{-jK\pi/2} - 1}{-jK2\pi} = \frac{1 - e^{-jK\pi/2}}{jK2\pi}$$

$$= \frac{e^{-jK\pi/4}}{jK2\pi}\left[e^{jK\pi/4} - e^{-jK\pi/4}\right] = \frac{e^{-jK\pi/4}}{K\pi}\left[\frac{e^{+jK\pi/4} - e^{-jK\pi/4}}{2j}\right]$$

$$= \frac{e^{-jK\pi/4}}{K\pi} \cdot \sin\left(\frac{K\pi}{4}\right) = e^{-jK\pi/4}\,\frac{\sin\left(\frac{K\pi}{4}\right)}{\left(\frac{K\pi}{4}\right)\cdot 4}$$

$$= \boxed{\frac{e^{-jK\pi/4}}{4}\,\text{sinc}\left(\frac{K}{4}\right) = c_K}$$

6. Assume $x(t)$ has the spectrum shown below (all angles are multiples of 45 degrees):



a) What is the ***average value*** of $x(t)$?

b) What is the ***average power*** in $x(t)$?

c) Write an expression for $x(t)$ in terms of cosines (Assume $\omega_0 = 1$). Your expression must be ***real***.

a) $\bar{x} = 3\angle 180^\circ = \boxed{-3 = \bar{x}}$

b) $P_x = |c_0|^2 + 2|c_1|^2 + 2|c_2|^2 + 2|c_3|^2 + 2|c_4|^2$

$= 3^2 + 2\cdot 2^2 + 2\cdot 1^2 = 9 + 8 + 2 = \boxed{19 = P_x}$

c) $x(t) = c_0 + \sum_{k=1}^{\infty} 2|c_k|\cos(k\omega_0 t + \angle c_k)$

$= -3 + 2\cdot 2\cos(kt - 90^\circ) + 2\cdot 1\cos(2kt + 45^\circ)$

$\boxed{x(t) = -3 + 4\cos(kt - 90^\circ) + 2\cos(2kt + 45^\circ)}$