4.0 System Properties

In this chapter we will start looking at various properties that can be used to characterize a system. We will initially illustrate these concepts as much as possible with examples from circuits and mechanical systems. However, since these are general concepts we will begin to explore abstract systems described only by algebraic, integral, or differential equations. Our goal is to be able to be able to determine whether or not a mathematical model of a system possesses these properties, and to develop the necessary vocabulary.

4.1 Linear (L) Systems

Let's assume we have a system with an input x(t) producing and output y(t). We can write this graphically as $x(t) \rightarrow y(t)$. A system possesses the *scaling* or *homogeneity* property if $\alpha x(t) \rightarrow \alpha y(t)$ for any constant α and any input $x(t)$. In particular, if a system possesses the homogeneity property, if the input is zero the output will be zero ($\alpha = 0$), if the input is doubled the output is doubled ($\alpha = 2$), and if we change the sign of the input we also change the sign of the output ($\alpha = -1$). These are very simple and common tests that can quickly be used to determine if a system does not possess the homogeneity property.

Next let's assume we name possible two inputs as $x_1(t)$ and $x_2(t)$, and we name the corresponding outputs $y_1(t)$ and $y_2(t)$. Hence we know $x_1(t) \rightarrow y_1(t)$ and $x_2(t) \rightarrow y_2(t)$. A system possesses the *additivity* property if

$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \rightarrow \alpha_1 y_1(t) + \alpha_2 y_2(t)$$

for all constants α_1 and α_2, and all inputs $x_1(t)$ and $x_2(t)$.

Definition: A *linear system* is any system that possesses both the homogeneity and the additivity properties.

Example 4.1.1. Consider the simple resistive circuit shown in Figure 4.1, with the system input defined as the input voltage $v^{in}(t)$ and the system output defined as the current flowing in the circuit, $i^{out}(t)$. For this simple system we have the mathematical model

$$i^{out}(t) = \frac{v^{in}(t)}{R}$$

Clearly this model satisfies the homogeneity requirement, since if we scale the input by a constant α we also scale the output by α,

$$\alpha\, i^{out}(t) = \frac{\alpha\, v^{in}(t)}{R}$$

Let's next assume we have two inputs $v_1^{in}(t)$ and $v_2^{in}(t)$, as shown in Figure 4.2. If we use superposition, we replace each voltage source with a short circuit and determine the resulting output current for each input voltage source acting alone. This gives us

$$i_1^{out}(t) = \frac{v_1^{in}(t)}{R} \text{ and } i_2^{out}(t) = \frac{v_2^{in}(t)}{R}$$

Adding these we clearly have

$$i^{out}(t) = i_1^{out}(t) + i_2^{out}(t) = \frac{v_1^{in}(t) + v_2^{in}(t)}{R} = \frac{v^{in}(t)}{R}$$

If should be clear for this example that if we scale both of the inputs, we would also scale both of the outputs, so

$$\alpha_1 i_1^{out}(t) + \alpha_2 i_2^{out}(t) = \frac{\alpha_1 v_1^{in}(t) + \alpha_2 v_2^{in}(t)}{R}$$

Since the system has both the property of homogeneity and additivity, the system is linear.

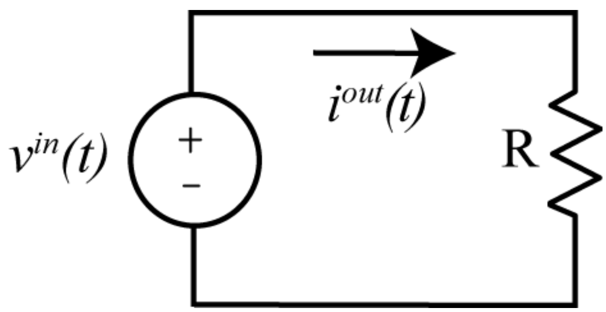

Figure 4.1. Circuit used in Example 4.1.1.

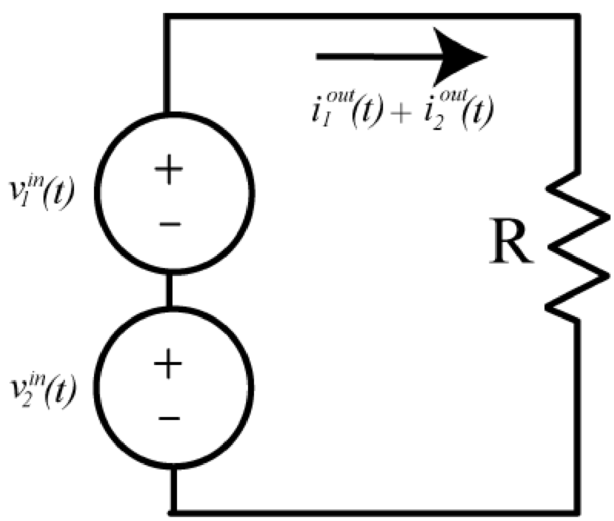

Figure 4.2. Second circuit used in Example 4.1.1.

Example 4.1.2. Consider the simple resistive circuit shown in Figure 4.3, with system input equal to the input voltage $v^{in}(t)$ and the system output equal to the voltage $v^{out}(t)$ measured across the resistor R_b. We assume the voltage $v_0 \neq 0$. The current flowing through the circuit is

$$i(t) = \frac{v^{in}(t) - v_0}{R_a + R_b}$$

and the output voltage is

$$v^{out}(t) = i(t)R_b = \left(\frac{v^{in}(t) - v_0}{R_a + R_b} \right) R_b$$

Now let's check to see if the homogeneity condition is satisfied. If the system input (the input voltage) is zero, $v^{in}(t) = 0$, then we expect the system output (the output voltage) to be zero. However, it is clear that under these circumstances the system output will be

$$v^{out}(t) = \frac{-v_0}{R_a + R_b} R_b \neq 0$$

Hence the homogeneity condition is not satisfied, and thus the system is not linear. It is important to note in this example that we need to look carefully at the system input and the system output.

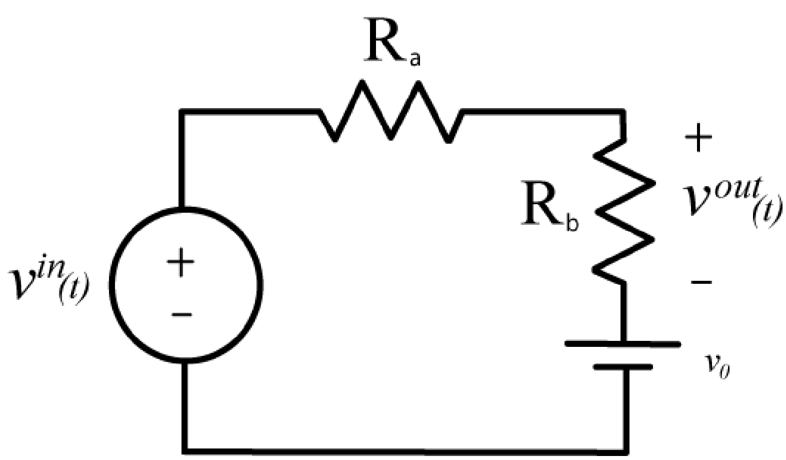

Figure 4.3. Circuit used in Example. 4.1.2.

Example 4.1.3. Consider the circuit shown in Figure 4.4. The system input is the input voltage $v^{in}(t)$ and the system output is the voltage across the capacitor, $v^{out}(t)$. The current flowing in the circuit is given by

$$i(t) = C \frac{dv^{out}(t)}{dt}$$

We then have

$$v^{in}(t) - \left(C \frac{dv^{out}(t)}{dt} \right) R = v^{out}(t)$$

or

$$\frac{dv^{out}(t)}{dt} + \frac{1}{RC}v^{out}(t) = \frac{1}{RC}v^{in}(t)$$

We can solve this using integrating factors, as before,

$$\frac{d}{dt}\left(v^{out}(t)e^{t/RC}\right) = v^{in}(t)e^{t/RC}$$

Next we integrate from some initial time t_0 up to the current time, t,

$$\int_{t_0}^{t} \frac{d}{d\lambda}\left(v^{out}(\lambda)e^{\lambda/RC}\right)d\lambda = \int_{t_o}^{t} v^{in}(\lambda)e^{\lambda/RC}d\lambda$$

$$v^{out}(t)e^{t/RC} - v^{out}(t_0)e^{t_0/RC} = \int_{t_0}^{t} v^{in}(\lambda)e^{\lambda/RC}d\lambda$$

Finally, we have the input-output relationship

$$v^{out}(t) = v^{out}(t_0)e^{-(t-t_0)/RC} + \int_{t_0}^{t} v^{in}(\lambda)e^{-(t-\lambda)}d\lambda$$

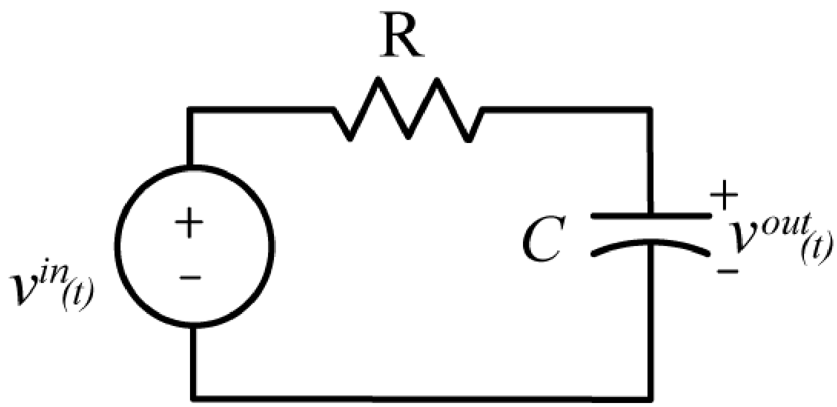

Figure 4.4. Circuit used in Example 4.1.3.

First let's check for homogeneity. If the input is zero, then we expect the output to be zero. However, in this case, if the input is zero, the output will be

$$v^{out}(t) = v^{out}(t_0)e^{-(t-t_0)/RC}$$

Hence, in order for the system to possess the homogeneity property, the initial conditions must be zero. ***<u>This is a general requirement for all systems.</u>*** Let's then assume the initial conditions are zero, then we have

$$v^{out}(t) = \int_{t_0}^{t} v^{in}(\lambda)e^{-(t-\lambda)}d\lambda$$

If we scale the input, we scale the output,

$$\left[\alpha v^{out}(t)\right] = \int_{t_0}^{t} \left[\alpha v^{in}(\lambda)\right] e^{-(t-\lambda)} d\lambda$$

Finally, if

$$v_1^{out}(t) = \int_{t_0}^{t} v_1^{in}(\lambda) e^{-(t-\lambda)} d\lambda$$

and

$$v_2^{out}(t) = \int_{t_0}^{t} v_2^{in}(\lambda) e^{-(t-\lambda)} d\lambda$$

then

$$\left[\alpha_1 v_1^{out}(t) + \alpha_2 v_2^{out}(t)\right] = \int_{t_0}^{t} \left[\alpha_1 v_1^{out}(t) + \alpha_2 v_2^{in}(\lambda)\right] e^{-(t-\lambda)} d\lambda$$

Hence the system also meets the additivity condition and is thus a linear system.

Example 4.1.4. The following models of systems, with system input $x(t)$**,** and system output $y(t)$**,** do not satisfy the homogeneity condition, and hence are not linear models:

$$y(t) = x(t) + 2$$
$$y(t) = |x(t)|$$
$$y(t) = e^{x(t)}$$
$$y(t) = \sin(x(t))$$
$$y(t) = \frac{1}{x(t)}$$

4.2 Testing for Linear Systems

We will present two different, though equivalent, methods for testing for linearity of a mathematical model of a system. A mathematical model of a system must pass one of these (or an equivalent) test for the model to be linear. It is much easier to show a mathematical model of a system is not linear than it is linear. The first method we will demonstrate assumes we have an algebraic or integral relationship between the system input and the system output. This test is straightforward, but we do not want to have to solve a differential equation to use it. Thus, we will present a second method to use with differential equations.

In order to simplify notation, we will assume we have a system operator, which we will denote as $\mathcal{H}$. The output of a system is the result of the system operator operating on the input. Hence if the system input is $x(t)$ and the system output is $y(t)$, then we would have

$$y(t) = \mathcal{H}\{x(t)\}$$

Consider the two signal flow diagrams shown in Figure 4.5. In the top figure, we examine the output of the system, $z_1(t)$, when the input to the system is the input $\alpha_1 x_1(t) + \alpha_2 x_2(t)$, $z_1(t) = \mathcal{H}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\}$. In the bottom figure, we examine the output of the system to input $x_1(t)$, $y_1(t) = \mathcal{H}\{x_1(t)\}$, and input $x_2(t)$, $y_2(t) = \mathcal{H}\{x_2(t)\}$, then form the linear combination of these, $z_2(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t) = \alpha_1 \mathcal{H}\{x_1(t)\} + \alpha_2 \mathcal{H}\{x_2(t)\}$. If the output is the same for both paths, i.e., if $z_1(t) = z_2(t)$, then the system is linear. If this is not true, then the system is not linear. Let's illustrate the method with a few examples. In the following examples, we assume the system input is $x(t)$ and the system output is $y(t)$.

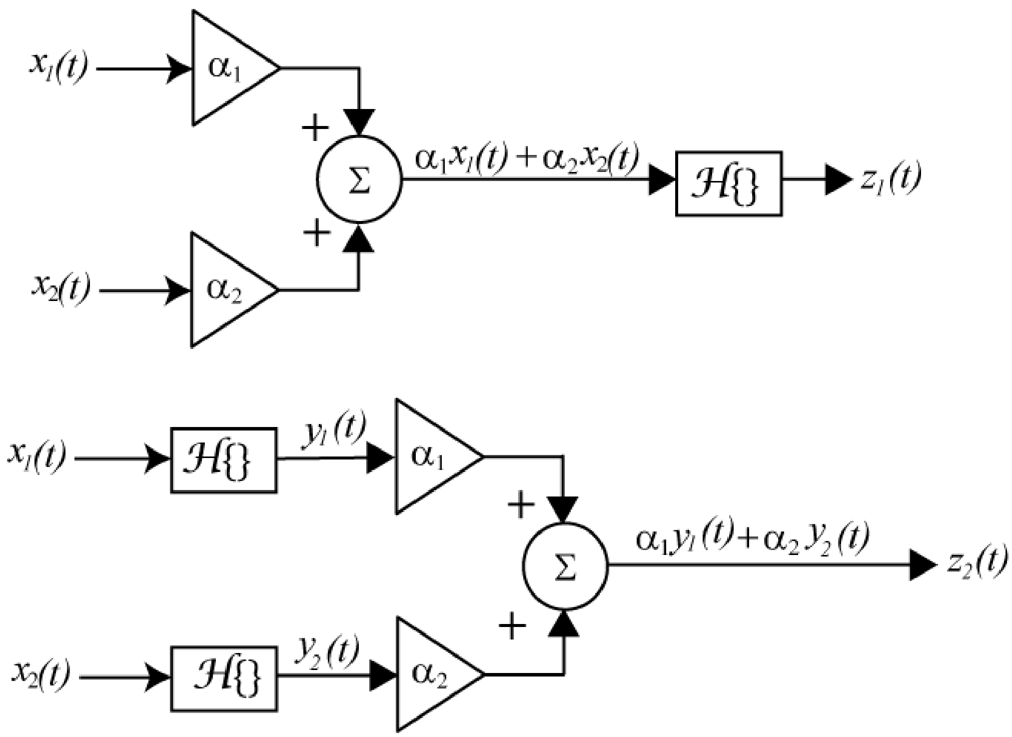

Figure 4.5. Signal flow graph of linearity test.

Example 4.2.1. Consider the mathematical model $y(t) = \sin^2(t)x(t)$. Does this represent a linear system? For this system, the linear operator is $\mathcal{H}\{x(t)\} = \sin^2(t)x(t)$. Along the top path we have

$$z_1(t) = \mathcal{H}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \sin^2(t)\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\}$$

Along the bottom path we have

$$z_2(t) = \alpha_1 \mathcal{H}\{x_1(t)\} + \alpha_2 \mathcal{H}\{x_2(t)\} = \alpha_1 \sin^2(t) x_1(t) + \alpha_2 \sin^2(t) x_2(t) = \sin^2(t)\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\}$$

Since $z_1(t) = z_2(t)$, the mathematical model is linear.

Example 4.2.2. Consider the mathematical model $y(t) = x(t) + b$**.** Does this represent a linear system? Without even really trying, we know this equation does not satisfy the homogeneity condition, so it does not represent a linear system. However, let's see what happens with our new test. For this system, the linear operator is $\mathcal{H}\{x(t)\} = x(t) + b$.

Along the top path we have

$$z_1(t) = \mathcal{H}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} + b$$

Along the bottom path we have

$$z_2(t) = \alpha_1 \mathcal{H}\{x_1(t)\} + \alpha_2 \mathcal{H}\{x_2(t)\} = \alpha_1\{x_1(t) + b\} + \alpha_2\{x_2(t) + b\} = \{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} + b\{\alpha_1 + \alpha_2\}$$

Now if we compare these, we do not have $z_1(t) = z_2(t)$ for all possible α_1 and α_2, hence the model is not linear.

Example 4.2.3. Consider the mathematical model $y(t) = \int_{-\infty}^{t} e^{-2(t-\lambda)} x(\lambda) d\lambda$. Does this represent a linear system? Along the top path we have

$$z_1(t) = \mathcal{H}\{\alpha_1 x_1(t) + \alpha_2 x_2(t)\} = \int_{-\infty}^{t} \{\alpha_1 x_1(\lambda) + \alpha_2 x_2(\lambda)\} d\lambda$$

Along the bottom path we have

$$z_2(t) = \alpha_1 \mathcal{H}\{x_1(t)\} + \alpha_2 \mathcal{H}\{x_2(t)\}$$

$$z_2(t) = \alpha_1 \int_{-\infty}^{t} e^{-2(t-\lambda)} x_1(\lambda) d\lambda + \alpha_2 \int_{-\infty}^{t} e^{-2(t-\lambda)} x_2(\lambda) d\lambda = \int_{-\infty}^{t} e^{-2(t-\lambda)} \{\alpha_1 x_1(\lambda) + \alpha_2 x_2(\lambda)\} d\lambda$$

Since $z_1(t) = z_2(t)$, the mathematical model is linear.

The signal flow graph method we have presented works well for determining if a mathematical model of a system is linear, provided we have the output of the system written as an algebraic or integral function of the input. However, it is very common to model systems in terms of differential equations, and we would like to be able to determine if a system modeled by a differential equation is linear without having to solve the differential equation, as fun as that might be.

The second method we will present for determining if a mathematical description of a system is linear is easier to demonstrate then explain, but the general idea is as follows:

1. Write two differential equations, one with system input $x_1(t)$ and system output $y_1(t)$, the second with system input $x_2(t)$ and system output $y_2(t)$.

2. Multiply the $x_1(t) \to y_1(t)$ equation by α_1 and the $x_2(t) \to y_2(t)$ equation by α_2

3. Add the equations together and regroup, we want to write the resulting differential equation in terms of $\alpha_1 x_1(t) + \alpha_2 x_2(t)$ and $\alpha_1 y_1(t) + \alpha_2 y_2(t)$.

4. Make the substitutions $X(t) = \alpha_1 x_1(t) + \alpha_2 x_2(t)$ and $Y(t) = \alpha_1 y_1(t) + \alpha_2 y_2(t)$ in the differential equation.

5. If the resulting differential equation is the same as the original differential equation, with $x(t)$ replaced by $X(t)$ and $y(t)$ replaced by $Y(t)$, then we have shown that
$$\alpha_1 x_1(t) + \alpha_2 x_2(t) \to \alpha_1 y_1(t) + \alpha_2 y_2(t)$$
and we can conclude that the system is linear. If this is not true, then the system is not linear.

Note that in order to satisfy the homogeneity conditions, we must assume the initial conditions for the system are all zero.

Example 4.2.4. Consider the mathematical model $\dot{y}(t) + \sin^2(t) y(t) = e^{-t} x(t)$**.** Does this represent a linear system? We have

$$\dot{y}_1(t) + \sin^2(t) y_1(t) = e^{-t} x_1(t) \quad \text{and} \quad \dot{y}_2(t) + \sin^2(t) y_2(t) = e^{-t} x_2(t)$$

Multiplying by α_1 and α_2 we have

$$\alpha_1 \dot{y}_1(t) + \alpha_1 \sin^2(t) y_1(t) = \alpha_1 e^{-t} x_1(t) \text{ and } \alpha_2 \dot{y}_2(t) + \alpha_2 \sin^2(t) y_2(t) = \alpha_2 e^{-t} x_2(t)$$

Adding and regrouping we have

$$\left[\alpha_1 \dot{y}_1(t) + \alpha_2 \dot{y}_2(t)\right] + \sin^2(t)\left[\alpha_1 y_1(t) + \alpha_2 y_2(t)\right] = e^{-t}\left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right]$$

Substituting we have

$$\dot{Y}(t) + \sin^2(t) Y(t) = e^{-t} X(t)$$

Thus, this system represents a linear system.

Example 4.2.5. Consider the mathematical model $\dot{y}(t) + y(t)x(t) = x(t)$. Does this represent a linear system? We have

$$\dot{y}_1(t) + y_1(t)x_1(t) = x_1(t) \text{ and } \dot{y}_2(t) + y_2(t)x_2(t) = x_2(t)$$

Multiplying by α_1 and α_2 we have

$$\alpha_1 \dot{y}_1(t) + \alpha_1 y_1(t)x_1(t) = \alpha_1 x_1(t) \text{ and } \alpha_2 \dot{y}_2(t) + \alpha_2 y_2(t)x_2(t) = \alpha_2 x_2(t)$$

Adding and regrouping we have

$$\left[\alpha_1 \dot{y}_1(t) + \alpha_2 \dot{y}_2(t)\right] + \alpha_1 y_1(t)x_1(t) + \alpha_2 y_2(t)x_2(t) = \left[\alpha_1 x_1(t) + \alpha_2 x_2(t)\right]$$

Substituting we have

$$\dot{Y}(t) + \alpha_1 y_1(t)x_1(t) + \alpha_2 y_2(t)x_2(t) = X(t)$$

At this points, it is clear that we cannot write this resulting differential equation just in terms of $X(t)$ and $Y(t)$, so the system is not linear.

Example 4.2.6. Consider the mathematical model $\dot{y}(t)+y(t)=x(t)+2$. Does this represent a linear system? This one is easy if we use the homogeneity condition. If the input is zero the output should also be zero. However, in this model, if the input is zero, we still have the output being nonzero. Hence this model is nonlinear.

We should point out that this technique can be used for systems that are not differential equations, but it may sometimes be more difficult than the flow-graph techniques.

4.2 Time-Invariant (TI) Systems

A time-invariant (TI) system is one in which, if the input $x(t)$ is delayed by an amount T then the output $y(t)$ is delayed by the same amount, without changing shape. Symbolically, if for a system we have $x(t) \rightarrow y(t)$, then if the system is also time-invariant we will have $x(t-T) \rightarrow y(t-T)$. Figure 4.6 presents a signal flow graph test for time-invariance, assuming we can write the output as an algebraic function or integral of the input. Along the top path we delay the input and then determine the output. Along the bottom path, we put the usual input into the system and then delay the usual output. If the results of these two paths are identical, then the system is time-invariant. There are a few subtleties involved in this test, so read through the following examples carefully.

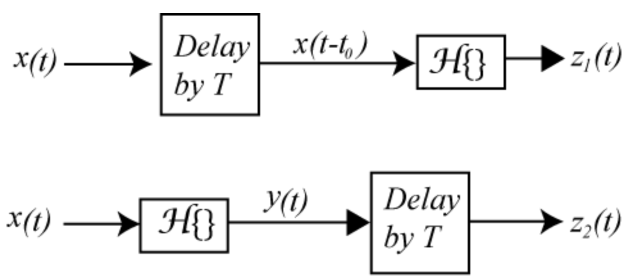

Figure 4.6. Signal flow graph of time invariance test.

Example 4.2.1. Consider a system with the mathematical model $y(t)=\alpha(t)x(t)$. Does this model represent a linear system? Along the top path, delaying the input we have

$$z_1(t)=\mathcal{H}\{x(t-T)\}=\alpha(t)x(t-T)$$

Along the second path (delaying the output) we have

$$z_2(t)=\left[\mathcal{H}\{x(t)\}\right]_{t=t-T}=\alpha(t-T)x(t-T)$$

Clearly $z_1(t)\neq z_2(t)$**,** so the model is not time-invariant.

Example 4.2.2. Consider the RC circuit shown in Figure 4.7, with an initial charge on the capacitor. The system input is the applied voltage $x(t)$ and the system output is the voltage across the capacitor $y(t)$. We need to write the output as a function of the input. The current in the loop is given by

$$\frac{x(t)-y(t)}{R} = C\frac{dy(t)}{dt}$$

or

$$\frac{dy(t)}{dt} + \frac{1}{RC}y(t) = \frac{1}{RC}x(t)$$

$$\frac{d}{dt}\left[y(t)e^{t/RC}\right] = e^{t/RC}x(t)$$

Integrating both sides and rearranging we get

$$\int_{t_0}^{t}\frac{d}{d\lambda}\left[y(\lambda)e^{\lambda/RC}\right]d\lambda = y(t)e^{t/RC} - y(t_0)e^{t_0/RC} = \int_{t_0}^{t} e^{\lambda/RC}x(\lambda)d\lambda$$

$$y(t) = y(t_0)e^{-(t-t_0)/RC} + \int_{t_0}^{t} e^{-(t-\lambda)/RC}x(\lambda)d\lambda$$

Along the top path of the signal flow graph, delaying the input, we have

$$z_1(t) = \mathcal{H}\left\{x(t-T)\right\} = y(t_0)e^{-(t-t_0)/RC} + \int_{t_0}^{t} e^{-(t-\lambda)/RC}x(\lambda-T)d\lambda$$

Note that we only delay the input, we do not change any other functions. Along the second path of the signal flow graph we only delay the output, which means we take the output and replace all instances of t with $t-T$. This leads to

$$z_2(t) = \left[\mathcal{H}\left\{x(t)\right\}\right]_{t=t-T} = y(t_0)e^{-(t-T-t_0)/RC} + \int_{t_0}^{t-T} e^{-(t-T-\lambda)/RC}x(\lambda)d\lambda$$

Now we want to see if $z_1(t) = z_2(t)$**.** Since we do not know what the input is, we will change variables in the integrals so both of them are simple functions of the dummy integral variable. $z_2(t)$ is already in the correct form, so we need to change variables in $z_1(t)$. Let's let $\sigma = \lambda - T$, or $\lambda = \sigma + T$. Then can rewrite $z_1(t)$ as

$$z_1(t) = y(t_0)e^{-(t-t_0)/RC} + \int_{t_0-T}^{t-T} e^{-(t-T-\sigma)/RC}x(\sigma)d\sigma$$

Now let's compare $z_1(t)$ and $z_2(t)$, and determine if they are equal or if we can make them equal. First of all, the terms associated with the initial conditions cannot be made equal, so if this system is going to be time invariant *the initial conditions must be zero.* Both of them have the same integrand, and both have the same upper limit on the integral. The only difference in the integral term is the lower limit. In order for the two integrals to be equal there are two choices. The most obvious choice is to assume that the input is zero for all times before the initial time t_0. Then the lower limit on the integral in $z_1(t)$ is effectively still effectively t_0,

$$\begin{aligned} z_1(t) &= \int_{t_0-T}^{t-T} e^{-(t-T-\sigma)/RC} x(\sigma) d\sigma \\ &= \underbrace{\int_{t_0-T}^{t_0} e^{-(t-T-\sigma)/RC} x(\sigma) d\sigma}_{x(\sigma)=0 \text{ in this range}} + \int_{t_0}^{t-T} e^{-(t-T-\sigma)/RC} x(\sigma) d\sigma \\ &= \int_{t_0}^{t-T} e^{-(t-T-\sigma)/RC} x(\sigma) d\sigma \end{aligned}$$

The other option is for $t_0 = -\infty$. With this choice of initial value, the first terms are equal and we have

$$z_1(t) = z_2(t) = \int_{-\infty}^{t-T} e^{-(t-T-\sigma)/RC} x(\sigma) d\sigma$$

Note that in order for a system to be time-invariant, the initial conditions must be zero, just as they are for the linearity requirements.

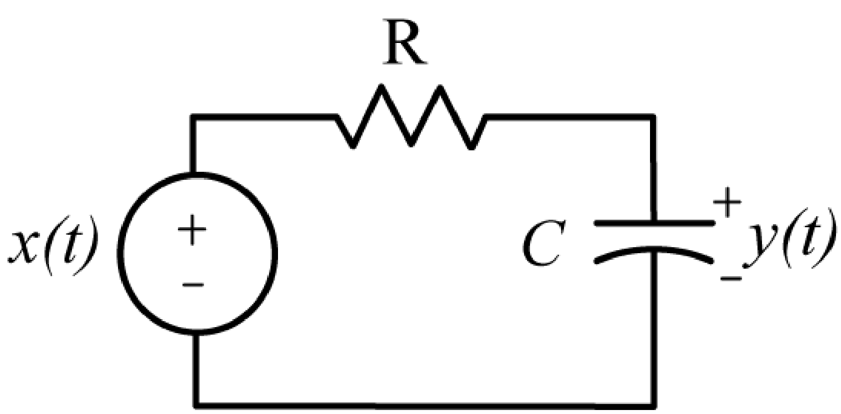

Figure 4.7. Circuit used in Example 4.2.2.

Example 4.2.3. Consider the model of a system $y(t) = x\left(\frac{t}{2}\right)$. Is this model time-invariant? This example is somewhat tricky, since we have to interpret the tests correctly. Along the top path, we are supposed to subtract T from the argument of $x(t)$, so we have

$$z_1(t) = \mathcal{H}\{x(t-T)\} = x\left(\frac{t}{2} - T\right)$$

This is not the result you expect, but it is the correct way to interpret the top path of our signal flow graph. Along the bottom path we delay the output by T, so we have

$$z_2(t) = \left[\mathcal{H}\left\{x(t)\right\}\right]_{t=t-T} = x\left(\frac{t-T}{2}\right)$$

Clearly $z_1(t) \neq z_2(t)$, so the model is not time-invariant.

Example 4.2.4. Consider the model of a system $y(t) = x(1-t)$. Is this model time-invariant? We have

$$z_1(t) = \mathcal{H}\left\{x(t-T)\right\} = x(1-t-T)$$

and

$$z_2(t) = \left[\mathcal{H}\left\{x(t)\right\}\right]_{t=t-T} = x(1-t+T)$$

Clearly $z_1(t) \neq z_2(t)$, so the model is not time-invariant.

Testing differential equations for time-invariance in general is somewhat more difficult, so we will just state a result: If the differential equation is just a function of the input $x(t)$ and the output $y(t)$, and these are both just simple functions of t (e.g. there are no terms like $x(2t), y(t/2), x(1-t)$), then the differential equation is time-invariant *if there are no other functions of time other than the input and output functions.*

Example 4.2.5. The following models of systems are not time invariant (though they are linear!):

$$\dot{y}(t) = e^t x(t)$$
$$\ddot{y} + \sin(t)\dot{y}(t) = \cos(t)x(t)$$
$$\dot{y}(t) + y(t) = tx(t)$$

The following models of systems are time-invariant:

$$\dot{y}(t) + x(t)y(t) = x(t)$$
$$\ddot{y}(t) + \dot{y}(t) = x(t)$$
$$\dot{y}(t) + y(t) = x^2(t)$$

4.3 Causal Systems

If a system is *causal*, then the system output $y(t_0)$ at some arbitrary time t_0 can only depend on the system input $x(t)$ up until (and including) time t_0. Another way of describing a causal system is that it is nonanticipative, the output does not anticipate the input. While most of the systems we think of are causal, this is because they work in real-time. That is, we are collecting or storing data to be processed as the data comes in, not at a later time. However, many discrete-time systems, such as an ipod or MP3 player, have data or music stored in advance. When the music is played it is possible for the system to look at future values of the input (the discrete-time signal representing the music) and make adjustments based on the "future". This is possible because these systems do not need to work in real-time. In what follows, we again assume the system input is $x(t)$, the system output is $y(t)$, and $x(t) \rightarrow y(t)$.

Example 4.3.1. Consider the mathematical model of a system $y(t) = x(t+1)$. Does this model represent a causal system? Often the easiest way to analyze problems like this is to put in various values of t. Thus for $t = 0$ we have $y(0) = x(1)$, and clearly the output at time zero depends on the input at time one, and the system is not causal.

Example 4.3.2. Consider the mathematical model of a system $y(t) = e^{(t+1)}x(t)$. Does this model represent a causal system? This is a causal system, since the output $y(t)$ at any time t depends only on the system input at the same time. Remember that we are only concerned with input-output relationships. The exponential term does not affect the causality of the system.

Example 4.3.3. Consider the mathematical model of a system $y(t) = x(-t)$. Does this model represent a causal system? This system is not causal, since for $t = -1$ we have $y(-1) = x(1)$, and the output depends on a future value of the input.

Example 4.3.4. Consider the mathematical model of a system $y(t) = x\left(1 - \frac{t}{2}\right)$. Does this model represent a causal system? This system is a bit more difficult to analyze than the previous systems. If this system is causal, then we must have $t \geq 1 - \frac{t}{2}$ or $\frac{3}{2}t \geq 1$. This will not be true for all time, so the system is not causal.

Example 4.3.5. Consider the mathematical model of a system $\dot{y}(t) + 2ty(t) = x(t+1)$. Does this model represent a causal system? In order to answer this, we will solve for the output as a function of the input (and review integration factors as an added bonus!). We have

$$\frac{d}{dt}\left[y(t)e^{t^2} \right] = \dot{y}(t)e^{t^2} + 2te^{t^2}y(t) = e^{t^2}[\dot{y}(t) + 2ty(t)] = e^{t^2}\left[x(t+1)\right]$$

or

$$\frac{d}{dt}\left[y(t)e^{t^2} \right] = e^{t^2}\left[x(t+1)\right]$$

Integrating both sides from some initial time t_0 up to an arbitrary final time t we have

$$\int_{t_0}^{t} \frac{d}{d\lambda}[y(\lambda)e^{\lambda^2}]d\lambda = y(t)e^{t^2} - y(t_0)e^{t_0^2} = \int_{t_0}^{t} e^{\lambda^2}x(\lambda+1)d\lambda$$

Rearranging this we have

$$y(t) = y(t_0)e^{-t^2+t_0^2} + \int_{t_0}^{t} e^{-t^2+\lambda^2} x(\lambda+1)d\lambda$$

Now we can determine that the output at any time t, $y(t)$, depends on the input from time t_0+1 up until time $t+1$. Thus the system is not causal.

4.4 Memoryless Systems

A system is *memoryless* or *instantaneous* if the output at any time t does not depend on past or future values of the input. If the output does directly depend on the input, then $y(t)$ must be an algebraic function of $x(t)$.

Example 4.4.1. Consider the mathematical model of a system $y(t) = x(t+1)$. Does this model represent a memoryless system? No, the system model is not memoryless, since the output at any time depends on a future input.

Example 4.4.2. . Consider the mathematical model of a system $y(t) = x(t-1)$. Does this model represent a memoryless system? No, the system model is not memoryless, since the output at any time depends on a past input.

Example 4.4.3. . Consider the mathematical model of a system $y(t) = x^2(t) + 2$. Does this model represent a memoryless system? Yes, the system model is memoryless, since the output at any time depends on the input only at that time.

Example 4.4..4 . Consider the mathematical model of a system $\dot{y}(t) + 2ty(t) = x(t+1)$. Does this model represent a memoryless system? No, the system model is not memoryless, since the output at any time depends on past and future values of the input.

4.5 Invertible Systems

An *invertible* system a system in which each output is associated with a unique input. That is, there is a one-to-one relationship between the system input and the system output.

Example 4.5.1. The mathematical models of systems $y(t) = \cos(x(t))$ and $y(t) = x^2(t)$ are not invertible, since there is more than one input that produces the same output.

4.6 Bounded Input Bounded Output (BIBO) Stable Systems

A mathematical model of a system that produces a bounded output for every bounded input is a *bounded-input bounded-output stable system*. Note that we do not need to know what the output is for every input, only that it is bounded. In addition, since we are looking at the input output relationship *we assume all of the initial conditions are zero*. Mathematically, if $|x(t)| \le M$ for some finite constant M means $|y(t)| < N$ for some finite constant N, then the system is BIBO stable.

Example 4.6.1. Is the mathematical model of a system $y(t) = e^{x(t)}$ BIBO stable? If we assume $|x(t)| \le M$ then we have $y(t) \le e^M = N$ and the model is BIBO stable.

Example 4.6.2. Is the mathematical model of a system $y(t) = \cos\left(\frac{1}{x(t)}\right)$ BIBO stable ?
The answer is yes, since we know that the cosine is always bounded between -1 and 1, so the output is always bounded, even if we do not know what it is.

Example 4.6.3. Is the mathematical model of a system $y(t) = \int_0^t e^{-(t-\lambda)} x(\lambda) d\lambda$ BIBO stable? Since we assume the input is bounded, $|x(t)| \le M$, we can write

$$y(t) = \int_0^t e^{-(t-\lambda)} x(\lambda) d\lambda \le M \int_0^t e^{-(t-\lambda)} d\lambda = Me^{-t} \int_0^t e^{\lambda} d\lambda = Me^{-t}(e^t - 1) = M(1 - e^{-t}) \le M$$

Hence $|y(t)| \le M$ and the output is bounded, so this is a BIBO stable model.

4.7 Linear Time-Invariant (LTI) Systems

In this course we will focus our attention on systems that are both linear and time-invariant, commonly referred to as LTI systems. If we have an LTI system and know the response of the system to specific input, then we can determine the response of the system to any linear and time shifted combination of those inputs. For example, assume we know the input/output relationships $x_i(t) \to y_i(t)$ for various inputs $x_i(t)$ and the corresponding outputs $y_i(t)$. Then, since the system is LTI, we know $\alpha_i x_i(t) \to \alpha_i y_i(t)$, $\sum_i \alpha_i x_i(t) \to \sum_i \alpha_i y_i(t)$, $x_i(t - t_i) \to y_i(t - t_i)$, and $\sum_i \alpha_i x_i(t - t_i) \to \sum_i \alpha_i y_i(t - t_i)$. Let's illustrate the implications of this with a few examples.

Example 4.7.1. Assume that we know that if the input is $x(t) = u(t)$, a unit step (Heaviside) function, then the output of an LTI system will be $y(t) = e^{-t}u(t)$. Now assume we want to use this information to determine the response of the system to a pulse of width (duration) T and amplitude A. The first thing we need to do is to write out new input $x_{new}(t)$ in terms of our known input. We can write the pulse input as

$$x_{new}(t) = Au(t) - Au(t - T) = Ax(t) - Ax(t - T)$$

Since the system is LTI we know

$$Ax(t) \rightarrow Ay(t)$$
$$Ax(t-T) \rightarrow Ay(t-T)$$

and the new output will be

$$y_{new}(t) = Ay(t) - Ay(t-T) = Ae^{-t}u(t) - Ae^{-(t-T)}u(t-T)$$

4.8 Linearizing Nonlinear System Models

While we will be focusing on linear systems, many systems we commonly use are actually nonlinear. However, if we operate them only over a limited range of inputs, the assumption of linearity is reasonably accurate. This is very common with electronic circuits, such as BJT and MOSFET transistors, where we use "small signal" models and assume these devices can be modeled as linear for small enough input signals.

In general, if we have an input-output relationship of the form, $y(t) = f(x(t))$, then we can use a Taylor series approximation of f that we can use for small $x(t)$. It may seem odd that we are allowing x to be a function of time, but the idea is the same. When the input is zero we have $y(0) = f(0)$, and this provides our nominal operating point. We can then approximate the output for small inputs as

$$y(t) \approx f(0) + \left[\frac{df(x)}{dx}|_{x=0} \right] x(t) = y(0) + \left[\frac{df(x)}{dx}|_{x=0} \right] x(t)$$

If we look at the deviations from the operating point, we have

$$y(t) - y(0) = \Delta y(t) \approx \left[\frac{df(x)}{dx}|_{x=0} \right] x(t) = m x(t)$$

where we have defined m as the slope near $x = 0$, $m = \frac{df(x)}{dx}|_{x=0}$. With this approximation we have the linear relationship between $\Delta y(t)$ and $x(t)$, $\Delta y(t) = mx(t)$.

Example 4.8.1. Consider the nonlinear system model $y(t) = \frac{1}{2 + x(t)}$ **.** Determine a linear model for small signals. We can write this as

$$y(0) = f(0) = \frac{1}{2}$$
$$y(t) = f(x) = (2+x)^{-1}$$

$$\frac{df(x)}{dx}|_{x=0} = -1(2+x)^{-2}|_{x=0} = -\frac{1}{4}$$

so

$$y(t) \approx \frac{1}{2} - \frac{1}{4}x(t)$$

Looking at the deviation about the operating point we get the small signal linear model

$$\Delta y(t) = y(t) - \frac{1}{2} \approx -\frac{1}{4}\text{x(t)}$$

Example 4.8.2. Consider the nonlinear system model $y(t) = 1 + 3e^{x(t)}$**.** Determine a linear model for small signals. We can write this as

$$y(t) = f(0) = 4$$
$$y(t) = f(x) = 1 + 3e^{x}$$
$$\frac{df(x)}{dx}\Big|_{x=0} = 3e^{x}\Big|_{x=0} = 3$$

so

$$y(t) \approx 4 + 3x(t)$$

We then have the small signal linear model

$$\Delta y(t) = y(t) - 4 = 3x(t)$$