Alfred R. Schmidt Freshman Mathematics Competition 2013

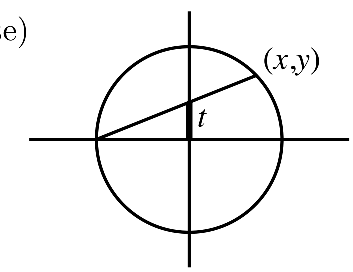
The answers are a very good start. You should also include as much of your reasoning as needed to convince anyone reading your write-up that your result is correct. Most of the credit will be earned for clear, concise, correct exposition. Including scratch-work is not the same as writing a clear explanation. If you are unable to completely justify your solution, partial solutions may be awarded some credit. Computers, calculators and other computing aids are not allowed. You may keep this copy of the questions.

1 The sum $S = 7^{2013} + 7^{2015} + 7^{1776} + 7^{1778}$ is computed. Then S is divided by one thousand and the remainder R is computed. Determine the value of R. Recall that calculators and computers are not allowed.

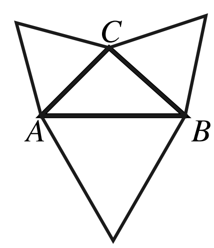
2 Think of 5, 7 and 6 as integers in decimal form. Express 5/7 base 6 as a number of the form $0.a_1a_2a_3a_4a_5a_6\ldots$.

3 Prove that $x^2 - x + 1$ divides $(x^4 + x^3 + x^2 + x + 1)^2 + (x^4 + x^3 + x^2 + x + 1) + 1$.

4 Let (x, y) be a point on the unit circle. Parametrize (write) x and y as functions of t.



5 Given triangle ABC. Construct equilateral triangles t_1, t_2, t_3 on the sides of triangle ABC. Let P_1, P_2, P_3 be the centers of t_1, t_2, t_3, respectively. Prove that triangle $P_1P_2P_3$ is an equilateral triangle



6 Express $\displaystyle\sum_{k=1}^{\infty} \frac{2^{k+2}}{(2^{k+1}-1)(2^k-1)}$ as a rational number.

Alfred R. Schmidt Freshman Mathematics Competition 2014

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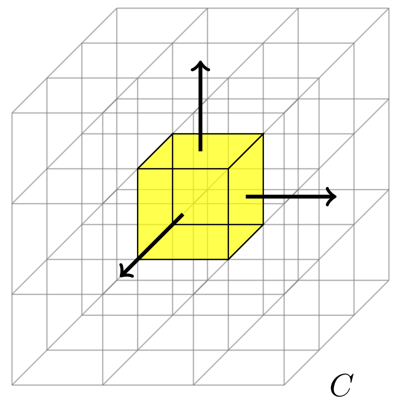
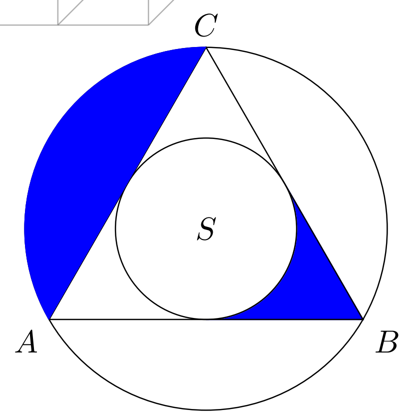
1 Let $\lfloor k \rfloor$ represent the greatest integer less than or equal to k. Compute $\sum_{k=1}^{100} \lfloor \sqrt{k} \rfloor$.

2 Twenty-seven cubes of cheese are packaged in a $3 \times 3 \times 3$ cube. To the chagrin of the cheesemaker, a worm has found its way to the center cube. Once the worm has eaten a cube it can move into any adjacent cube whose face was touching the former. So the worm can move North, South, East, West, Up, and Down but not diagonally. Can the worm eat all the cheese?



3 Determine the value of the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3 - 1}{n^3 + 1}.$$



4 Equilateral triangle ABC is inscribed in a circle and circle S is inscribed in triangle ABC. Prove that the area of the shaded region is the same as the area of circle S.

5 Find all values of x so that

$$\cos(x)\cos(2x)\cos(4x) = \frac{1}{8}.$$

6 Quadrilateral $ABCD$ has area 2014. The midpoints of each side $PQRS$ are drawn as shown. Prove that $PQRS$ is a parallelogram and determine the area of $PQRS$.