Final Project Due 20 November 2019 at 5:00 PM

Complete Tasks 1 and 2 individually. Complete Task 3 with your lab team

1. One state linear optimal control by Gauss PS.

Solve the following linear finite horizon optimal control problem using the Gauss-Legendre PS method presented in class, and compare to the indirect solution. Use 10 collocation points for the Gauss PS solution and at least 100 points for the indirect solution. The cost function is:

$$J = \frac{1}{2}\int_{t_0}^{t_f} (x(t)^T \cdot Q \cdot x(t) + u(t)^T \cdot R \cdot u(t))dt \tag{1}$$

The state dynamics are

$$\frac{dx}{dt} = A \cdot x(t) + B \cdot u(t) \tag{2}$$

With boundary conditions:

$$x(t_0) = 1; \quad x(t_f) = 0 \tag{3}$$

Set $A = B = Q = R = 1$. Find the indirect solution by forming the Hamiltonian system:

$$\frac{d}{dt}\begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} = \begin{bmatrix} A & -B \cdot R^{-1} \cdot B^T \\ -Q & -A^T \end{bmatrix} \cdot \begin{bmatrix} x(t) \\ \lambda(t) \end{bmatrix} \tag{4}$$

and solving using the Burchett method presented early in the quarter. Show that the solution is:

$$\begin{gathered} x(t) = 1.0000 \cdot e^{-\sqrt{2}t} - 7.2135 \times 10^{-7} \cdot e^{\sqrt{2}t} \\ \lambda(t) = 2.4124 \cdot e^{-\sqrt{2}t} - 2.9879 \times 10^{-7} \cdot e^{\sqrt{2}t} \\ u(t) = -\lambda(t) \end{gathered} \tag{5}$$

Plot both solutions of the time history of the state on one axis. Plot both solutions of the control on another axis.

2. One state non-linear optimal control by Gauss PS.

Complete the non-linear example presented in class, and plot the known solution. Use 10 collocation points for the Gauss PS solution and at least 100 points for the known solution. The cost function is:

$$J = \frac{1}{2}\int_{t_0}^{t_f} (q \cdot y(t) + r \cdot u(t)^2)dt \tag{6}$$

The state dynamics are

$$\frac{dy}{dt} = 2 \cdot a \cdot y(t) + 2 \cdot b \cdot \sqrt{y(t)} \cdot u(t) \tag{7}$$

With boundary conditions:

$$y(t_0) = 2; \quad y(t_f) = 1 \tag{8}$$

Set $a = b = q = r = 1$. The known solution is:

$$y(t) = \left(1.4134 \cdot e^{-\sqrt{2}t} + 8.4831 \times 10^{-4} \cdot e^{\sqrt{2}t}\right)^2 \tag{9}$$

3. The caber toss.

You don't need to physically toss a caber to pass this assignment, but you will compute the optimal trajectory and minimum force required under certain assumptions. The caber toss is an event of the Scottish highland games where competitors throw a pole weighing as much as 200 lbs. A successful throw causes the caber to rotate 270 degrees from the starting vertical position and 'point' directly away from the competitor in its final horizontal position. In order to achieve this, the competitor lifts the proximal end of the caber to waist height, balancing the caber as an inverted pendulum. He then walks forward, allowing the caber to begin rotating forward to the launch angle. Finally, he heaves mostly upward as the distal end of the caber rotates downward to achieve enough height and angular velocity such that the caber lands near a vertical position after 180 degrees of total rotation. See for instance this video of world champion Dan McKim (https://www.youtube.com/watch?v=xb0FU8rSisU) .

In order to determine the smallest force and optimal trajectory necessary for a successful toss you will solve a multiple phase problem using Gauss PS. The first phase begins with the caber moving slowly forward at the launch angle and zero angular velocity. Constant forces F_x and F_y are applied to the proximal end of the caber. (See below for axis definitions.) The dynamic constraints can be written from conservation of linear and angular momentum as:

$$\frac{d^2x}{dt^2} = -g + \frac{F_x}{m} \tag{10}$$

$$\frac{d^2y}{dt^2} = \frac{F_y}{m} \tag{11}$$

$$I\frac{d^2\theta}{dt^2} = \frac{L}{2}F_x \sin(\theta) - \frac{L}{2}F_y \cos(\theta) \tag{12}$$

The second phase begins when the contestant loses contact with the caber ($F_x = F_y = 0$) and ends when the distal end of the caber strikes the ground. You can use Eqs. (10)-(12) for this phase as well, setting the contact forces to zero. When the distal end strikes the ground, the angular velocity is instantaneously reduced by the inelastic collision. The angular velocity after the collision can be calculated as

$$\left.\frac{d\theta}{dt}\right|_{2n+1} = \left(I\left.\frac{d\theta}{dt}\right|_{2n} + m\frac{L}{2}\sin(\theta|_{2n})\left.\frac{dx}{dt}\right|_{2n} - m\frac{L}{2}\cos(\theta|_{2n})\left.\frac{dy}{dt}\right|_{2n}\right)\Big/\left(I + \frac{mL^2}{4}\right) \tag{12}$$

In order for the caber to reach vertical and tumble away from the competitor, the minimum angular velocity after striking the ground is given by conservation of energy

$$\left.\frac{d\theta}{dt}\right|_{2n+1} \geq \sqrt{\frac{3g}{L}\left(1+\cos(\theta|_{2n})\right)} \tag{13}$$

Equations (12) and (13) can be combined into a single boundary condition on the states at the end of phase two

$$\sqrt{\frac{3g}{L}\left(1+\cos(\theta|_{2n})\right)} = \left(I\left.\frac{d\theta}{dt}\right|_{2n} + m\frac{L}{2}\sin(\theta|_{2n})\left.\frac{dx}{dt}\right|_{2n} - m\frac{L}{2}\cos(\theta|_{2n})\left.\frac{dy}{dt}\right|_{2n}\right)\Big/\left(I+\frac{mL^2}{4}\right) \tag{14}$$

Thus, your non-linear constraint function must include instances of Eqs. (10)-(12) for each phase (put these in state space form, so you will actually have six equations for each phase), and Eq. (14) for the final boundary condition. You will also need the following geometric constraints at the initial and final states:

$$x|_1 = \left(1 + \frac{L}{2}\cos\,(\theta|_1)\right) \tag{15}$$

$$y|_1 = \left(\frac{L}{2}\sin(\theta|_1)\right) \tag{16}$$

$$x|_{2n} = -\left(\frac{L}{2}\cos(\theta|_{2n})\right) \quad \text{, since } \cos(\theta|_{2n}) < 0 \tag{17}$$

Linear constraints can be used to enforce the initial conditions

$$\dot{x}|_1 = 0 \tag{18}$$

$$\dot{y}|_1 = 2.0 \tag{19}$$

Assume non-zero forward velocity at toss initiation.

$$\dot{\theta}\big|_1 = 0 \tag{20}$$

Finally, in order to stitch the two phases together, the states at the end of phase one must match the states at the beginning of phase two

$$x|_n = x|_{n+1};\;\; \dot{x}|_n = \dot{x}|_{n+1} \tag{21}$$

$$y|_n = y|_{n+1};\;\; \dot{y}|_n = \dot{y}|_{n+1} \tag{22}$$

$$\theta|_n = \theta|_{n+1};\;\; \dot{\theta}|_n = \dot{\theta}|_{n+1} \tag{23}$$

In order to give fmincon a reasonable chance of finding a solution, assume phase one takes one second, and phase two takes one second. Use the following parameters:

$$m = 75\,kg;\ \ g = 9.81\frac{\mathrm{m}}{\mathrm{s}^2};\ \ \ L = 3\,m \tag{23}$$

HINTS:

- use 12 collocation points for each phase.
- Use the following non-linear geometric constraints on the initial conditions:

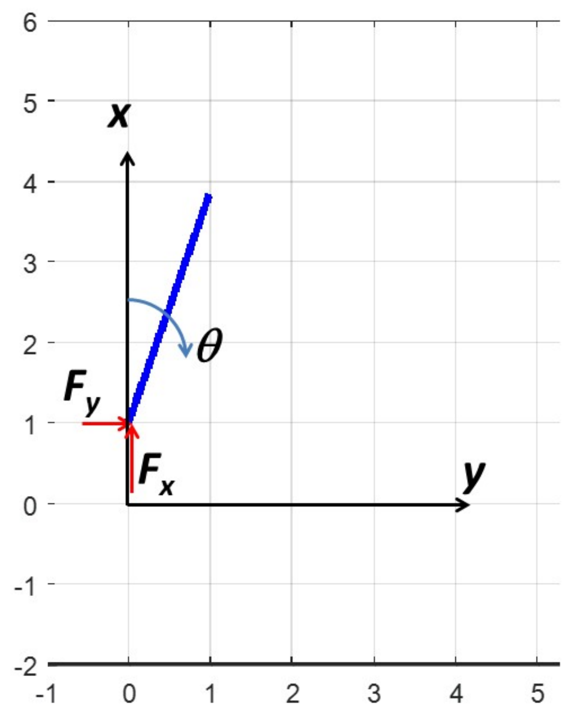
Proximal end is 1m above ground, cg is L/2cosθ above proximal end:

$$x_1 - 1 - \frac{L}{2}\cos\theta_1 = 0$$

$$y_1 - \frac{L}{2}\sin\theta_1 = 0$$

Distal end strikes ground at end of phase 2:
(cosθ< 0 after nearly π rad of rotation)

$$x_{2N} + \frac{L}{2}\cos\theta_{2N} = 0$$



- Use upper and lower bounds to help fmincon find the solution:

$$-\infty < \dot{x} < \infty$$
$$0 < \dot{y} < \infty$$
$$0 < x < \infty$$
$$0 < y < \infty$$
$$0 < \theta < \pi$$
$$0 < \dot{\theta} < \infty$$