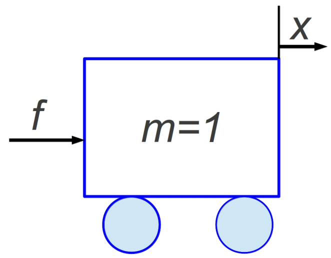
ROSE-HULMAN INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

ME 506 Advanced Controls

Homework 6, Due 22 October 2019

1. <u>Solution to a simple optimal control problem:</u> Consider the problem of moving a mass on a frictionless surface from an initial point x_0 to a desired point x_f.



The equation of motion is simply:

$$m\ddot{x} = f(t) \tag{1}$$

Convert this second order equation to a matrix set of first order equations in the form:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}f \tag{2}$$

Where . I.E.

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} & \\ & \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} \\ \end{bmatrix} f \tag{3}$$

You will write a set of functions to integrate Eqn. 3 using 4th order Runge Kutta.

The 'optimal' control force $f(t)$ to return the mass to zero from an arbitrary starting point can be calculated as

$$f(t) = -\mathbf{R}^{-1}\mathbf{B}^T\mathbf{p}(t)$$

Where $\mathbf{p}(t)$ is given from:

$$\mathbf{p}(t) = [\mathbf{\Theta}_{21} + \mathbf{\Theta}_{22}\mathbf{S}]\,[\mathbf{\Theta}_{11} + \mathbf{\Theta}_{12}\mathbf{S}]^{-1}\,\mathbf{x}(t)$$

And the $\mathbf{\Theta}_{ij}$ terms are given as

$$\left[\begin{array}{c|c} \mathbf{\Theta}_{11} & \mathbf{\Theta}_{12} \\ \hline \mathbf{\Theta}_{21} & \mathbf{\Theta}_{22} \end{array}\right] = \left[\begin{array}{cc|cc} 1 & -\frac{1}{2}e^{-T} + \frac{1}{2}e^{T} & -\frac{1}{2}e^{-T} + \frac{1}{2}e^{T} - T & -\frac{1}{2}e^{-T} - \frac{1}{2}e^{T} + 1 \\ 0 & \frac{1}{2}e^{-T} + \frac{1}{2}e^{T} & \frac{1}{2}e^{-T} + \frac{1}{2}e^{T} - 1 & \frac{1}{2}e^{-T} - \frac{1}{2}e^{T} \\ \hline 0 & 0 & 1 & 0 \\ 0 & \frac{1}{2}e^{-T} - \frac{1}{2}e^{T} & \frac{1}{2}e^{-T} - \frac{1}{2}e^{T} & \frac{1}{2}e^{-T} + \frac{1}{2}e^{T} \end{array}\right]$$

Where $T = t - t_f$, $\mathbf{R} = 1$, and finally:

$$\mathbf{S} = \begin{bmatrix} 75 & 0 \\ 0 & 1 \end{bmatrix}$$

1. <u>Write a set of Matlab functions that integrate Eqn 3 using Runge-Kutta 4.</u>
   a. Write a helper function that finds the value of the control based on current time, final time and current state.

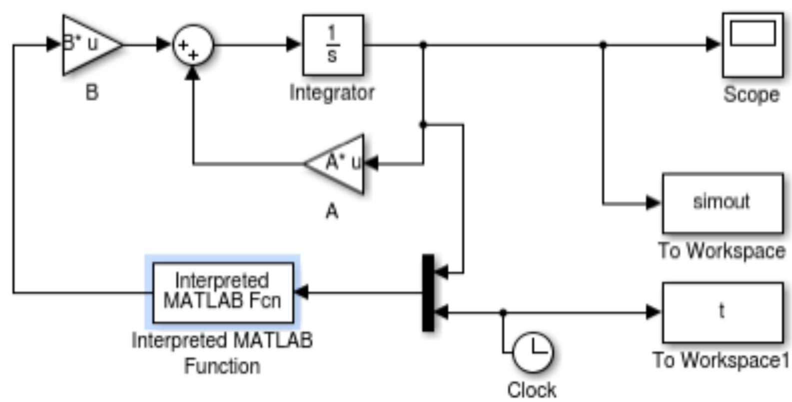
b. Write a helper function that uses part a and the current state to compute the state derivatives.

c. Write a top level that integrates Equation 3 from $t = 0$ to $t_f = 5.5$ with an initial condition of $\mathbf{x} = [1\ 0]^T$. Use a time step of 0.01. Store the state history in a 551x2 matrix where the first row is position and the second is velocity. You may use **ode45** or an equivalent ode solver for this.

d. Plot the position, and velocity of the mass, and the control effort $f(t)$ as functions of time.

e.

2. Prepare a simulink model as shown below. Note that the embedded Matlab function needs to compute p using the current time as its only input. In this case, p is the transformation from state to co-state (not the co-state) such that the co-state = $p{\cdot}x(t)$. The blocks labeled 'A' and 'B' and the integrator are part of the plant and will be replaced with the plant interface block in lab.



Part 3: add damping

Suppose the system model is modified to include damping such that the state-space is:

$$\begin{Bmatrix} \dot{x} \\ \ddot{x} \end{Bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -c \end{bmatrix} \begin{Bmatrix} x \\ \dot{x} \end{Bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} f \tag{3}$$

Use Maple or symbolic Matlab to determine a symbolic form of the state transition matrix ($\boldsymbol{\Theta}$), knowing that

$$\left[\begin{array}{c|c} \boldsymbol{\Theta}_{11} & \boldsymbol{\Theta}_{12} \\ \hline \boldsymbol{\Theta}_{21} & \boldsymbol{\Theta}_{22} \end{array}\right] = \exp\left(\begin{bmatrix} \mathbf{A}T & -\mathbf{B}\mathbf{R}^{-1}\mathbf{B}^T T \\ -\mathbf{Q}T & -\mathbf{A}^T T \end{bmatrix}\right) \tag{3}$$