Reconstruction of Partially Conductive Cracks Using Boundary Data

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Abstract

This paper develops and algorithm for finding one or more non-insulated, pair-wise disjoint, linear cracks in a two-dimensional region using boundary measurements.

1 Introduction

In the field of non-destructive testing of materials, the ability to view the inside of an object based solely on boundary measurements proves invaluable. This paper is concerned with the identification of one or more cracks inside a homogeneous object. The technique we use is impedance or steady-state thermal imaging. In [4], Alessandrini and Diaz offer uniqueness results for the identification of n cracks inside a two-dimensional region, and in [1] Trainor and Krieger employed the reciprocity gap formula to identify linear, perfectly insulating cracks. In [3], Ogborne and Vellela deal with the case of a single partially conducting linear crack and examine the nature of the ill-posedness of its associated inverse problem. In this paper, we expand the ideas of [1] and [3] to find a method to locate a single partially conductive linear crack, and we then move on to develop an algorithm for finding multiple non-insulated cracks.

2 The Forward Problem

Let Ω be a bounded region in $\mathbb{R}^2$ with boundary $\partial\Omega$; we will use coordinates $\mathbf{x} = (x_1, x_2)$. Suppose that within Ω there exist n pairwise disjoint line segment "cracks". Let σ_i denote the i^{th} crack, $\mathbf{p}_i$ the midpoint of σ_i, $|\sigma_i|$ the length of σ_i, and θ_i the angle between the line containing σ_i and the x-axis, where $-\frac{\pi}{2} < \theta_i \leq \frac{\pi}{2}$. We use $\Sigma = \cup_{i=1}^{n} \sigma_i$ to denote the collection of cracks. We assume in general that $|\sigma_i| \ll \min\{|\mathbf{x} - \mathbf{p}_i|; \mathbf{x} \in \partial\Omega\}$.

Suppose we apply a steady-state heat flux g to the boundary of $\partial\Omega$. Let $u(x_1, x_2)$ be the resulting steady-state temperature at any point inside of Ω. After appropriate scaling, we suppose that u satisfies the equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = 0 \text{ on } \Omega \backslash \Sigma \tag{1}$$

with the Neumann boundary condition

$$\frac{\partial u}{\partial \mathbf{n}} = g \text{ on } \partial\Omega, \tag{2}$$

where $\mathbf{n}$ is the unit outward normal vector to $\partial\Omega$.

We also need boundary conditions on the cracks. For all i, we will denote one side of σ_i as the "$-$" side, and the other side as the "$+$" side and choose a unit normal vector $\mathbf{n}_{\sigma_i}$ that points from the $-$ to the $+$ side. The superscript "$+$" will denote the limiting value of a quantity from the $+$ side of σ_i; likewise, the superscript "$-$" will denote the limiting value from the $-$ side of σ_i.

To model the fact that heat may flow over the crack, but with some resistance, we assume

$$\frac{\partial u}{\partial \mathbf{n}_{\sigma_\mathbf{i}}} \quad = \quad k[u] \quad \text{on } \sigma_i, \tag{3}$$

for all i, where $[u] = u^+ - u^-$ (in physical terms [u] is the temperature jump over the crack) and k is a constant. Note that (3) is assumed to hold on both sides of σ_i. Equation (3) models the crack as a kind of "contact resistance", with heat flowing over σ_i at a rate proportional to $[u]$. We also add the normalization $\int\limits_{\partial\Omega} u \; ds = 0$, which ensures a unique solution to the problem (1)-(3).

The solution u to our forward problem will be smooth away from the cracks (since u is harmonic there), but generally will have a jump discontinuity across each crack. Standard elliptic regularity theory shows that $[u]$ is continuous along any given crack and tapers to zero at the crack endpoints.

Given our conditions (1)-(3), the forward problem consists of determining $u(\mathbf{x})$ for each $\mathbf{x} \in \Omega \setminus \Sigma$, given the input flux g and knowledge of the crack locations Σ. However, we are interested in the inverse problem: We consider the cracks σ_i to be unknown and we are to recover their locations given that (1)-(3) hold and additional knowledge, namely measurements of u on $\partial\Omega$; note that we consider the constant k as known. Physically, this corresponds to applying a known heat flux g to the object Ω, measuring the steady-state temperature response on the outer boundary, and then inferring the location of any internal crack(s), given that we know the constitutive law that governs heat flow over the cracks.

3 Locating a Single Crack

In this section we consider the problem of locating a single crack, and proceed in three steps: First, we find the line on which the crack lies; next, we determine where on this line the center of the crack lies; finally, we determine the length of the crack.

3.1 Finding the Crack Line

Suppose that there exist no cracks in Ω. Let

$$\Gamma(x_1, x_2) = \frac{1}{4\pi} \ln(x_1^2 + x_2^2)$$

be the fundamental solution for the steady-state heat equation in two dimensions. Then by Green's Third Identity we have

$$\frac{1}{2}u(\mathbf{x}) + \int_{\partial\Omega} u(\mathbf{y}) \frac{\partial \Gamma}{\partial \mathbf{n_y}}(\mathbf{y}-\mathbf{x})\, dx_y - \int_{\partial\Omega} \Gamma(\mathbf{y}-\mathbf{x}) \frac{\partial u}{\partial \mathbf{n_y}}(\mathbf{y})\, ds_y \quad = \quad 0 \tag{4}$$

where $\mathbf{y} = (y_1, y_2)$ and ds_y refers to integration with respect to arc length on $\partial\Omega$ in the $\mathbf{y}$ variable. Let the left side of (4) be denoted by $\alpha(\mathbf{x})$. Note that we can compute α for any $\mathbf{x} \in \partial\Omega$ because everything in α is evaluated from the boundary data.

Now suppose there are n perfectly insulating cracks inside Ω. Then as shown in [1] (which applies the Divergence Theorem) we obtain

$$\alpha(\mathbf{x}) \quad = \quad \sum_{i=1}^{n} \int_{\sigma_i} \frac{\partial \Gamma}{\partial \mathbf{n}_{\sigma_i}}(\mathbf{x}-\mathbf{y})[u](\mathbf{y})\, ds_y. \tag{5}$$

We note here, however, that precisely the same argument applies when we have the boundary condition $\frac{\partial u}{\partial \mathbf{n}} = k[u]$ for $k \geq 0$; all that's really needed is a boundary condition which induces a jump over the cracks. As in the perfectly insulating case, our goal is to use the value of $\alpha(\mathbf{x})$ for each $\mathbf{x} \in \partial\Omega$ to determine the cracks.

For the rest of this section we will assume $n = 1$ and neglect the subscript on σ. In this case we have

$$\alpha(\mathbf{x}) \quad = \quad \int_{\sigma} \frac{\partial \Gamma}{\partial \mathbf{n}_{\sigma}}(\mathbf{x}-\mathbf{y})[u](\mathbf{y})\, ds_y. \tag{6}$$

We parameterize the crack σ as

$$\left\{ (y_1, y_2) : y_1 = p_1 + t\cos\theta, y_2 = p_2 + t\sin\theta \mid -\frac{L}{2} \leq t \leq \frac{L}{2} \right\}$$

in which $\mathbf{p} = (p_1, p_2)$ denotes the crack center, θ the crack angle, and L the crack length. In [1], they were able to calculate $\frac{\partial \Gamma}{\partial \mathbf{n}_\sigma}(\mathbf{x}-\mathbf{y})$ as

$$\frac{\partial \Gamma}{\partial \mathbf{n}_\sigma}(\mathbf{x}-\mathbf{y}) \quad = \quad \frac{1}{2\pi}\frac{(\mathbf{x}-\mathbf{p}, \mathbf{n}_\sigma)}{(\mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p}) - 2t(\mathbf{x}-\mathbf{p}, \hat{\sigma}) + t^2},$$

where $\hat{\sigma} = \langle \cos\theta, \sin\theta \rangle$ and $\frac{-L}{2} \le t \le \frac{L}{2}$, and we use the notation $(\mathbf{x}, \mathbf{y})$ to denote the usual inner product on $\mathbb{R}^2$. Inserting this into equation (6) shows that $\alpha(\mathbf{x}) = 0$ exactly when $(\mathbf{x}-\mathbf{p}, \mathbf{n}_\sigma) = 0$, so $\alpha(\mathbf{x})$ equals 0 when $\mathbf{x}-\mathbf{p}$ is perpendicular to $\mathbf{n}_\sigma$. Therefore, there exist at least two points, $\mathbf{a}, \mathbf{b} \in \partial\Omega$ such that $\alpha(\mathbf{a}) = \alpha(\mathbf{b}) = 0$. It follows that σ is contained in the line from $\mathbf{a}$ to $\mathbf{b}$. Therefore we know the line on which the crack lies and can find the angle. In [1], they were dealing with perfectly insulated cracks, but applying the results of [3] to our situation reveals that the same method works for partially conducting cracks.

3.2 Finding the Midpoint

To show how to obtain the midpoint of our crack, we again adapt methods developed in [1]. If we assume that the length of the crack is much less than the distance from our midpoint $\mathbf{p}$ to any $\mathbf{x} \in \partial\Omega$, then we can approximate $\frac{\partial \Gamma}{\partial \mathbf{n}_\mathbf{y}}$ by

$$\frac{\partial \Gamma}{\partial \mathbf{n}_\mathbf{y}} \approx \frac{1}{2\pi}\frac{(\mathbf{x}-\mathbf{p}, \mathbf{n}_\sigma)}{(\mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p})} \tag{7}$$

(since $|t|$ is small, so the corresponding terms in $\frac{\partial \Gamma}{\partial \mathbf{n}_\mathbf{y}}$ are negligible). Thus we can approximate (6) by

$$\alpha(\mathbf{x}) \approx \frac{1}{2\pi}\frac{(\mathbf{x}-\mathbf{p}, \mathbf{n}_\sigma)}{(\mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p})}\int_\sigma [u](\mathbf{y})ds_y = \frac{1}{2\pi}J\frac{(\mathbf{x}-\mathbf{p}, \mathbf{n}_\sigma)}{(\mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p})}, \tag{8}$$

where $J = \int_\sigma [u](\mathbf{y})ds_y$.

We now note that the jump integral $J = \int_\sigma [u](\mathbf{y})ds_y$ can be computed very easily, as shown in [2] and [3], once the crack line (in particular, the angle θ) is known. Let $\phi(x, y) = -\sin(\theta)x + \cos(\theta)y$, where θ is the angle of the crack with respect to the x-axis (which we found in Section 3.1). Then we have from the reciprocity gap formula that

$$\int_{\sigma} [u](\mathbf{y}) ds_y = \int_{\partial\Omega} \left(u \frac{\partial \phi}{\partial \mathbf{n}} - \phi \frac{\partial u}{\partial \mathbf{n}} \right) ds \tag{9}$$

Since the right-hand side of this equation is computable from boundary data, we can find J.

Since it was shown in [3] that J is "generically" non-zero, we can say that

$$\frac{\alpha(\mathbf{x})}{J} \approx \frac{1}{2\pi} \frac{(\mathbf{x}-\mathbf{p}, \mathbf{n}_\sigma)}{(\mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p})} \tag{10}$$

for any $\mathbf{x} \in \partial\Omega$. When $\mathbf{x}-\mathbf{p}$ is parallel to $\mathbf{n}_\sigma$, we have $(\mathbf{x}-\mathbf{p}, \mathbf{n}_\sigma) = |\mathbf{x}-\mathbf{p}|$, thus turning equation (10) into $\frac{\alpha(\mathbf{x})}{J} \approx \frac{1}{2\pi} \frac{(\mathbf{x}-\mathbf{p}, \mathbf{n}_\sigma)}{(\mathbf{x}-\mathbf{p}, \mathbf{x}-\mathbf{p})} = \frac{1}{2\pi} \frac{1}{|\mathbf{x}-\mathbf{p}|}$. Note that if $\mathbf{x}-\mathbf{p}$ were antiparallel to $\mathbf{n}_\sigma$, then the quantity would be negated.

Suppose $\mathbf{x_1}, \mathbf{x_2} \in \partial\Omega$ such that the line $(\mathbf{x_1}, \mathbf{x_2})$ is perpendicular to the line $(\mathbf{a}, \mathbf{b})$, which we found while recovering the crack angle. Also suppose that the vector $\mathbf{x_1}-\mathbf{p}$ is parallel to $\mathbf{n}_\sigma$, which implies that $\mathbf{x_2}-\mathbf{p}$ is antiparallel to $\mathbf{n}_\sigma$. Then (10) and the triangle inequality imply that

$$|\mathbf{x_1}-\mathbf{x_2}| \leq |\mathbf{x_1}-\mathbf{p}| + |\mathbf{p}-\mathbf{x_2}| = \frac{1}{2\pi} \left(\frac{J}{\alpha(\mathbf{x_1})} - \frac{J}{\alpha(\mathbf{x_2})} \right), \tag{11}$$

where the equality holds only when $\mathbf{p} \in (\mathbf{x_1}, \mathbf{x_2})$. Therefore we can find $\mathbf{p}$ by taking pairs of points perpendicular to the (now known) line containing σ and choosing the specific pair, $(\mathbf{x_1}, \mathbf{x_2})$, for which the equality holds in (11) as in Figure 1. We finally get $\mathbf{p}$ as

$$\mathbf{p} = (\mathbf{x_1}, \mathbf{x_2}) \cap (\mathbf{a}, \mathbf{b}). \tag{12}$$

We want to mention that because equation (5) always holds, we do not have to make the assumption that $\frac{\partial u}{\partial \mathbf{n}_\sigma} = k[u]$, but instead we could have

$$\frac{\partial u}{\partial \mathbf{n}_{\sigma_\mathbf{i}}} = F([u])$$

for any function F (at least, any F for which the boundary value problem is solvable).

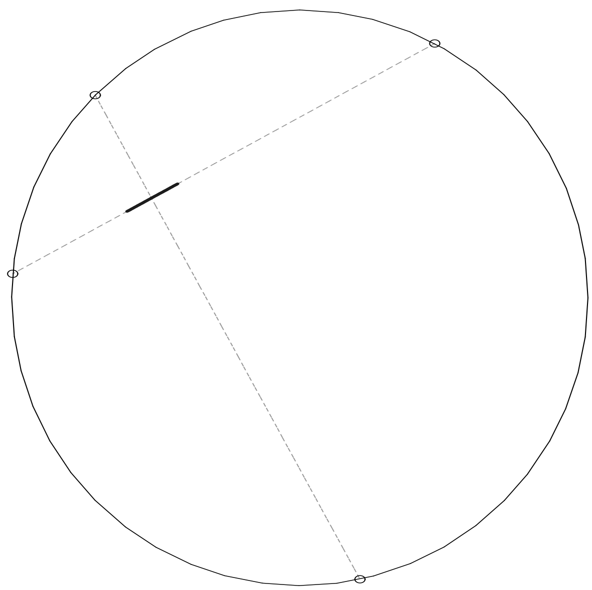

Figure 1: Method for locating the midpoint of the crack.

3.3 Finding the Length

In the last section we prove the following Theorem (or at least a special case of it), an extension of Theorem 1 in Section 5 of [1].

Theorem 3.1 *Let σ be a linear crack with center $\mathbf{p}$, at angle θ, of length L. Let u be the solution to the boundary value problem (1)-(3). Then*

$$\int_{\sigma}[u]\,ds = \frac{\pi/4}{1+\frac{8kL}{3\pi}}(\nabla u_0(\mathbf{p})\cdot\mathbf{n})L^2 + O(L^3) \tag{13}$$

where $O(L^3)$ denotes a quantity bounded in magnitude by CL^3 for all L sufficiently close to zero, with C independent of L (but C may depend on $\mathbf{p},\theta$, and the input flux g).

In the case $k = 0$ this becomes precisely Theorem 1 in Section 5 of [1]. Theorem 3.1 also holds for the multiple crack case.

The technique for finding the length consists in dropping the $O(L^3)$ term in equation (13), computing $J = \int_{\sigma}[u]\,ds$ from the boundary data, and then solving for L in equation (13) as

$$L \;=\; \frac{\frac{8}{3\pi}Jk + \sqrt{(Jk\frac{8}{3\pi})^2 + J\pi(\mathbf{n}\cdot\nabla u_0(\mathbf{p})}}{\frac{\pi}{2}(\mathbf{n}\cdot\nabla u_0(\mathbf{p}))} \tag{14}$$

where u_0 is the harmonic function on Ω with the same Neumann boundary data as u (which we can compute from the Neumann data g). Note that at the stage in which we are solving for L, we have already computed the crack line, hence we know $\mathbf{n}$, and we have also computed $\mathbf{p}$, so everything on the right in equation (14) is indeed known.

3.4 Example

We will now provide a specific example in which we locate a single crack. For the sake of simplicity we will take our domain Ω to be the unit circle in $\mathbb{R}^2$ with the conditions described in Section 2. We specify the heat flux on $\partial\Omega$ to be $g = \sin(t)$ and parameterize $\partial\Omega$ as $(\cos(t), \sin(t))$, for $0 \leq t < 2\pi$. We can write $u(\mathbf{x}) = u(t)$ where $\mathbf{x} = (\cos(t), \sin(t))$. We also have that the harmonic solution over Ω is

$$\begin{aligned} u_0(x,y) &= y, \\ \nabla u_0 &= \langle 0, 1 \rangle, \\ \int\limits_{\partial\Omega} u_0(\mathbf{y}) \frac{\partial \Gamma}{\partial \mathbf{n}_y}(\mathbf{y}-\mathbf{x})\, ds_y &= 0, \text{ and} \\ \int\limits_{\sigma} \Gamma(\mathbf{x}-\mathbf{y}) \frac{\partial u_0}{\partial \mathbf{n}}(\mathbf{y})\, ds_y &= \frac{1}{2}\sin t \end{aligned}$$

for the point $\mathbf{x} = (\cos t, \sin t)$. Thus in this case we obtain the simple expression

$$\alpha(\mathbf{x}) = \frac{1}{2}\left(u(t) - \sin t\right)$$

for $\mathbf{x} \in \partial\Omega$ and $0 \leq t < 2\pi$.

In the following example, we used a C program to generate the boundary data, $u(\mathbf{x})$, using a boundary integral approach for solving the boundary value problem. The program generates data for n equally spaced points on $\partial\Omega$.

Now we are ready to locate a single crack in Ω using the methods described in the previous sections We calculated boundary data for a unit circle with a crack described by a left endpoint at $(-0.1, 0.8)$, length $|\sigma| = 0.4$, and angle $\theta = -0.1$. We also fixed $k = 10$.

We can compute $\alpha(t)$ for each of our points on the boundary, and can see that $\alpha(t_1) = \alpha(t_2) = 0$ for $t_1 \approx 0.802899$ and $t_2 \approx 2.137376$. Therefore a line containing σ passes through the points

$$\begin{aligned} \mathbf{a} &= (\cos(0.802899), \sin(0.802899)) = (0.694624, 0.709373), \text{ and} \\ \mathbf{b} &= (\cos(2.137376), \sin(2.137376)) = (-0.536749, 0.843742) \end{aligned}$$

We now know the line containing σ, and can therefore compute the crack angle θ by taking the arctangent of the slope of this line. In this example, we calculate a value of $\theta \approx -0.100659$ (which is very close to the actual value, $\theta = -0.1$).

We can now use equation (9) to determine the value of the jump integral, which we will then use to find both the midpoint of the crack, and the crack length. First we will compute the integral over the m points on $\partial\Omega$,

$$\begin{aligned} \int_\sigma [u](\mathbf{y})\ ds_y &\approx \sum_{i=1}^{m} u_i \sin\left(\frac{2\pi(i-1)}{m} - \theta\right) \\ &- \int_0^{2\pi} \sin\phi\,(-\sin\theta\cos\phi + \cos\theta\sin\phi)\ d\phi \\ &= 0.029935 \end{aligned}$$

so that $J = \int_\sigma [u](\mathbf{y})\ ds_y \approx 0.029935$.

We can use this to help us locate the midpoint of the crack. Just as in [1], we test pairs of points on $\partial\Omega$ that define lines perpendicular to the crack line $(\mathbf{a}, \mathbf{b})$ that we just computed. To determine which points will help describe the midpoint of σ, we will use the test function

$$T(\mathbf{x_1}, \mathbf{x_2}) = \frac{1}{2\pi}\left(\frac{J}{\alpha(\mathbf{x_1})} - \frac{J}{\alpha(\mathbf{x_2})}\right) - |\mathbf{x_1} - \mathbf{x_2}|,$$

from [1]. The roots of this function, called $\mathbf{x_1}$ and $\mathbf{x_2}$, will define a line through $\mathbf{p}$, the midpoint of σ. Since $\mathbf{p} = (\mathbf{x_1}, \mathbf{x_2}) \cap (\mathbf{a}, \mathbf{b})$, our example gives us

$$\mathbf{p} \approx (0.041996, 0.785288)$$

After finding the length we can calculate the location of the left endpoint of the calculated crack and compare it to our initial value. This will come out to be at $(-0.165495, 0.806242)$. This is very near the true value $(-0.1, 0.8)$. Now having found both the angle and the midpoint of the crack, all we have left is to determine the crack's length.

We have computed J, we can compute $\mathbf{n}_\sigma \cdot \nabla u_0(\mathbf{p})$ (since we know $\mathbf{n}_\sigma$ and $\mathbf{p}$), and we know k. Using equation (14) for $|\sigma|$ gives us

$$|\sigma| \approx 0.417033.$$

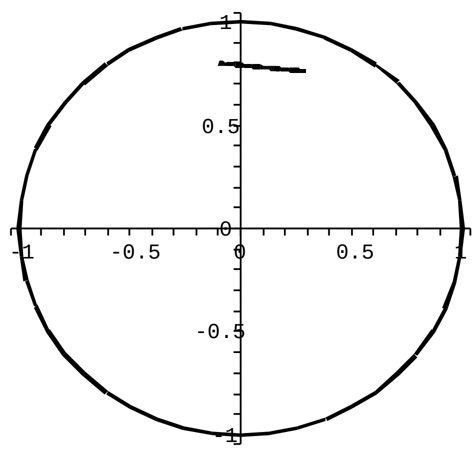

Figure 2: Reconstruction of a single crack where $k = 10$.

The algorithm we used gave us good results. Our true crack (with $k = 10$) was located with a left endpoint at $(-0.1, 0.8)$ with length $|\sigma| = 0.4$ and angle $\theta = -0.1$. Our calculated crack had a left endpoint (-0.165465 0.806242), $\theta \approx -0.100659$, and $|\sigma| \approx 0.417033$. As you can see from Figure 2, the true and computed cracks overlap almost exactly.

4 Locating Multiple Cracks

Finding multiple cracks poses a much greater challenge than locating a single one. We could not find any analytical way to solve the inverse problem for n cracks, and had to rely on an approximate numerical method to locate them, based on a least-squares approach. If we assume that the cracks are small and well-separated, then equation (5) becomes

$$\alpha(\mathbf{x}) \approx \frac{1}{2\pi} \sum_{i=1}^{n} J_i \frac{(\mathbf{x} - \mathbf{p}_i, \mathbf{n}_{\sigma_i})}{(\mathbf{x} - \mathbf{p}_i, \mathbf{x} - \mathbf{p}_i)} \tag{15}$$

We will write $\alpha(\mathbf{x}, \{\mathbf{p_i}, \theta_i, J_i\})$ to show the dependence of α on these parameters and use $\alpha^*(\mathbf{x})$ to denote the "true" value of α derived from measured data from a set of "true" cracks. Define

$$F(\mathbf{x}) \quad = \quad \alpha(\mathbf{x}, \{\mathbf{p_i}, \theta_i, J_i\}) - \alpha^*(\mathbf{x})$$

for $\mathbf{x} \in \partial\Omega$.

Since we could not find an analytical solution to finding multiple cracks, we will instead use a least-squares method to locate them. Specifically, we will make an initial guess at the parameters of each crack, then use measured data to compute $\alpha^*(\mathbf{x})$ and our guesses to compute $\alpha(\mathbf{x})$. Once the initial guess has been made, the Levenberg-Marquardt Method is used to tweak the parameters until $\sum_{j=1}^{m} F(\mathbf{x}_j)^2$ is minimized ($\mathbf{x_j}$ denotes the jth point of a list of m data points on $\partial\Omega$).

We would like for $F(\mathbf{x_j})$ to be zero for each $\mathbf{x_j} \in \partial\Omega$ when the correct parameters have been found for each crack. Usually this doesn't happen, but we do end up with a nice approximation for $\mathbf{p_i}$, θ_i, and J_i. Because of the jump formula 14, we can approximate each L_i. We should note that since our actual cracks are assumed to be well separated, the initial guess should have its cracks well separated also. We should also mention that this method has two substantial drawbacks: because it is purely numerical we cannot give an exact solution, and using this method pre-supposes that we know the number of cracks we are looking for.

It's worth noting that although we are using a least-squares approach, because we work with the function $\alpha(\mathbf{x})$ and use the analytic approximation in equation (15), we do NOT have to solve any forward problems. The computation of the objective function is very quick and easy.

4.1 Multiple Crack Examples

We chose the locations for two cracks and generated data for u on $\partial\Omega$, just as in the single crack example. For simplicity, we let both cracks have the same constant k. All other conditions are identical to those of the single crack case. We use a C program that employs the Levenberg-Marquardt algorithm, our least-squares approach to approximating the crack locations. At this point we make an initial estimate for the endpoint of each crack (we must know the number of cracks for this to work), the value of each jump integral, and the crack angles. If we have no idea where the cracks are located, we can make an arbitrary initial estimate. However, it may take a few attempts to find a guess.

First we let $k = 10$, and choose two cracks in Ω. We make initial estimates for the location of the cracks, and in this case the code gave us results after only 10 iterations. As you can see from Figure 3, the reconstruction is nearly perfect.

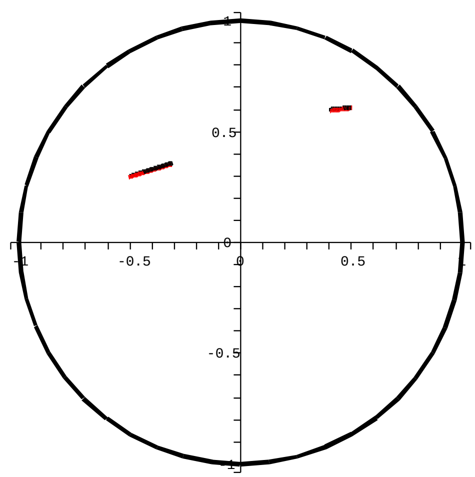

Figure 3: Reconstruction of two linear cracks with constant $k = 10$.

The algorithm becomes less stable as n (the number of cracks) increases. When we had three cracks (each with $k = 10$) in Ω, it took 50 iterations for the program to converge in one case. The resulting reconstruction, as shown in Figure 4, is not quite as accurate as with fewer cracks, but it is still a good approximation.

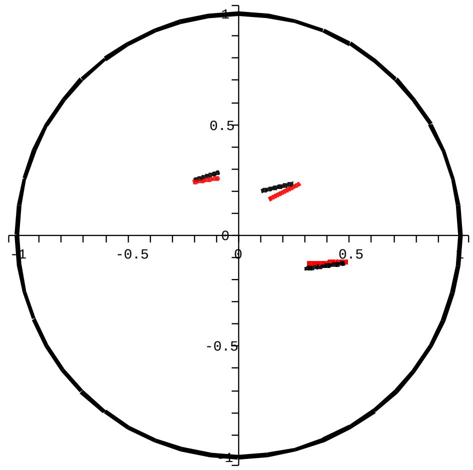

Figure 4: Reconstruction of three linear cracks with constant $k = 10$.

5 Finding Crack Length

In this section we provide a proof of a special case of Theorem 3.1. We suppose that Ω contains only a single line segment crack σ. Let u satisfy $\Delta u = 0$ in $\Omega\backslash\sigma$. We seek a function v, harmonic on $\Omega\backslash\sigma$, such that $u \approx u_0+v$, to good approximation. In this case we'd have $\frac{\partial u_0}{\partial \mathbf{n}} + \frac{\partial v}{\partial \mathbf{n}} = k[u_0 + v]$ on σ, leading to the requirement

$$\frac{\partial v}{\partial \mathbf{n}}(x) - k[v](\mathbf{x}) \quad = \quad -\frac{\partial u_0}{\partial \mathbf{n}}(\mathbf{x}),$$

at any point $\mathbf{x}$ on the crack.

We may rescale the coordinate system so that $\sigma = [0, 1]$ on the x_1-axis. If σ is short we may approximate $-\frac{\partial u_0}{\partial n}(x_1) \approx -\frac{\partial u_0}{\partial n}(\mathbf{p})$ where p is the midpoint of σ. Our boundary conditions are $\frac{\partial u}{\partial \mathbf{n}} = k[u]$ on σ (note that $\mathbf{n} = \langle 0, 1\rangle$ on σ), $\frac{\partial u}{\partial \mathbf{n}} = g$ on $\partial\Omega$. Let u_0 be the harmonic function on Ω with the same Neumann data g (u_0 is the solution on the "uncracked" domain). We will prove Theorem 3.1 in the special case that $\frac{\partial u_0}{\partial y}$ is constant on σ, and in fact equal to 1. A slightly more difficult argument, similar to that in [1], shows how to deal with the case in which $\frac{\partial u_0}{\partial y}$ is not constant.

First note that we can write $v \approx \frac{\partial u_0}{\partial \mathbf{n}}(\mathbf{p})w$ where w satisfies

$$\frac{\partial w}{\partial y}(x,0) - k[w](x) \;\; = \;\; -1 \qquad (16)$$

for $0 < x < 1$, since $[u_0] = 0$. We will construct a function w which satisfies equation (16) and is harmonic on $\mathbb{R}^2 \setminus [0,1]$. However, the corresponding approximation $u_0 + v$ to u will not satisfy the boundary condition on $\partial\Omega$—it will have slightly incorrect Neumann boundary data. However, the discrepancy can be shown to be small enough that $[u] = [v] + O(L^3)$, as shown in [1]. Thus the approximation will be sufficient for the purposes of proving Theorem 3.1.

Let us define the function $\phi(z) = 1 - 2z + 2z\sqrt{1 - \frac{1}{z}}$. It's easy to verify that ϕ is analytic on $\mathbb{C} \setminus [0,1]$. Let $v_j(x,y) = \operatorname{Im}(\phi^{j+1}(x+iy))$ for $j \geq 0$; the function v_j is harmonic on $\mathbb{R}^2 \setminus [0,1]$ and from [3] we know that

$$[v_j](x) \;\; = \;\; 4(-1)^j U_j(2x-1)\sqrt{x - x^2} \qquad (17)$$

$$\frac{\partial v_j}{\partial y}(x,0) \;\; = \;\; 2(j+1)(-1)^{j+1} U_j(2x-1) \qquad (18)$$

for $x \in (0,1)$, where U_j is the j^{th} Chebyshev polynomial of the second kind, and once again we use the bracket notation $[v_j]$ to denote the jump across the crack. We will try to construct a solution to equation (16) which decays rapidly away from σ, in the form

$$v(x,y) = \sum_{j=0}^{\infty} c_j v_j(x,y)$$

for appropriate constants c_j, $j \geq 0$.

Equations (16)-(18) lead us to

$$2\sum_{j=0}^{\infty}(-1)^{j+1}c_j\left((j+1)U_j(2x-1) + 2kU_k(2x-1)\sqrt{x-x^2}\right) \;\; = \;\; -1. \qquad (19)$$

We can multiply equation (19) by $U_m(2x-1)\sqrt{x-x^2}$, for $m \geq 0$, and integrate over $(0,1)$ to obtain

$$\frac{\pi}{4}(-1)^{m+1}(m+1)c_m \;\; + \;\; 4k\sum_{j=0}^{\infty}(-1)^{j+1}c_j \int_0^1 U_j(2x-1)U_m(2x-1)(x-x^2)\,dx$$

$$= \quad -\int_0^1 U_m(2x-1)\sqrt{x-x^2}\,dx,$$

an infinite system of equations in c_j, $j \geq 0$. This can be written as

$$\frac{\pi}{4}(-1)^{m+1}(m+1)c_m + 4k\sum_{j=0}^{\infty}(-1)^{j+1}B_{jm}c_j = a_m \tag{20}$$

for each $m \geq 0$, where $B_{jm} = \int_0^1 U_j(2x-1)U_m(2x-1)(x-x^2)\,dx$ for $j+m$ even and $B_{jm} = 0$ for $j+m$ odd, and $a_0 = -\frac{\pi}{8}$, $a_m = 0$ for $m \geq 1$.

More generally, consider equation (20) for arbitrary right side a_m. We can really write the whole thing as

$$\frac{\pi}{4}(m+1)c_m + 4k\sum_{j=0}^{\infty}B_{jm}c_j = (-1)^{m+1}a_m \tag{21}$$

using the fact that $B_{jm} = 0$ if $j+m$ is odd, as we show below. Indeed, we can work out an explicit formula for the B_{jm}, as follows. We have to look at

$$B_{jm} = \int_0^1 U_j(2x-1)U_m(2x-1)(x-x^2)\,dx.$$

Using a change of variables and the fact that $U_j(\cos t) = \frac{\sin((j+1)t)}{\sin t}$ (which is just a property of the Chebyshev polynomials) we get

$$B_{jm} = \frac{1}{8}\int_0^{\pi} \sin((j+1)t)\sin((m+1)t)\sin t\,dt. \tag{22}$$

If we work the integral, we find that if $j+m$ is odd we get zero, and if $j+m$ is even we get

$$B_{jm} = \frac{(j+1)(m+1)}{2(j+m+1)(j+m+3)(j-m+1)(m-j+1)}.$$

The formula for B_{jm} when $j+m$ is even is a straightforward consequence of standard integration techniques.

Note it is easy to see $B_{jm} < 0$ if $j \neq m$. If $j = m$ we get

$$B_{mm} = \frac{(m+1)^2}{2(2m+1)(2m+3)}.$$

Finally, let us divide through equation (21) by $\frac{\pi}{4}(m+1)$ to obtain

$$c_m + \frac{16k}{\pi(m+1)} \sum_{j=0}^{\infty} B_{jm} c_j = (-1)^{m+1} \frac{4}{\pi(m+1)} a_m := \tilde{a}_m \tag{23}$$

The system (23) can thus be written as $\mathbf{Ac} = \tilde{\mathbf{a}}$, where $\mathbf{c} = (c_0, c_1, \ldots)$. We will consider $\mathbf{c}$ and $\tilde{\mathbf{a}}$ to be sequences in the space ℓ^∞ and $\mathbf{A}$ as the linear operator from ℓ^∞ to ℓ^∞ defined by

$$(\mathbf{Ac})_m = \sum_{j=1}^{\infty} A_{mj} c_j$$

where $A_{mj} = \delta_j^m + \frac{16k}{\pi(m+1)} B_{jm}$ (here δ_m^j is the Kronecker delta). We will prove unique solvability of $\mathbf{Ac} = \tilde{\mathbf{a}}$ for any $\tilde{\mathbf{a}} \in \ell^\infty$ by showing that $\mathbf{A}$ is diagonally dominant and using a Neumann series inversion. We will then use this to show that our original problem (with $a_0 = -\pi/8$) is solvable.

For diagonal dominance we need to show that $\left| \frac{16k}{\pi(m+1)} \sum_{j \neq m} B_{jm} \right| < c \left(1 + \frac{16k}{\pi(m+1)} \right)$ for some c with $0 \leq c < 1$, i.e.,

$$- \sum_{0 \leq j \leq \infty, j \neq m} B_{jm} \leq c \left(\frac{\pi}{16k}(m+1) + B_{mm} \right) \tag{24}$$

for all $k \geq 0$, and all m, for some constant $0 \leq c < 1$, where we have used $B_{jm} \leq 0$ for $j \neq m$. Now the case $k = 0$ is trivial so we may assume $k > 0$.

We can calculate that if m is even, then

$$\sum_{j=0, j \text{even}}^{\infty} B_{jm} = \frac{1}{8(m+1)}.$$

If m is odd then

$$\sum_{j=0, j \text{edd}}^{\infty} B_{jm} = \frac{m+1}{8m(m+2)}.$$

Equation (24) would then require that $-\left(\sum_j B_{jm} - B_{mm}\right) \leq c\left(B_{mm} + \pi\frac{m+1}{16k}\right)$, or, if m is even,

$$\frac{4m^3+8m^2+4m+1}{8(m+1)(2m+1)(2m+3)} \leq c\left(\frac{m+1}{2(2m+1)(2m+3)} + \frac{\pi(m+1)}{16k}\right).$$

A bit of rearrangement yields

$$\frac{1}{c} \leq \frac{4(m+1)^3}{4m^3+8m^2+4m+1} + \frac{\pi(m+1)^2(2m+1)(2m+3)}{2k(4m^3+8m^2+4m+1)}. \tag{25}$$

For any fixed $k > 0$, we want the value of c closest to zero (but less than 1) that makes equation (25) true for all even m. It is easy to see that the first term on the right in equation (25) is always bigger than 1 (since $4(m+1)^3 = 4m^3 + 12m^2 + 12m + 4$), but approaches one as m approaches infinity. The second term is strictly increasing for $m \geq 0$ (this is each to check by plotting it as a function of m). It attains it's minimum value of $\frac{\pi}{2k}$ at $m = 0$. We can thus conclude that equation (25) is satisfied for $m \geq 0$ if we take c such that $\frac{1}{c} \leq 1 + \frac{\pi}{2k}$, or $c \geq 1/(1 + \frac{\pi}{2k})$. We thus take

$$c = \frac{1}{1+\frac{\pi}{2k}} = \frac{2k}{2k+\pi}. \tag{26}$$

Note $c < 1$.

In the case that m is odd, the diagonal dominance condition (24) becomes

$$\frac{(m+1)(4m^3+8m^2-3)}{8m(m+2)(2m+1)(2m+3)} \leq c\left(\frac{(m+1)^2}{2(2m+1)(2m+3)} + \frac{\pi(m+1)}{16k}\right).$$

A bit of algebra gives

$$\frac{1}{c} \leq \frac{4m(m+1)(m+2)}{4m^3+8m^2-3} + \frac{\pi m(m+2)(2m+1)(2m+3)}{2k(4m^3+8m^2-3)}. \tag{27}$$

It's again easy to check that the first term is greater than 1 for all ODD $m \geq 1$. Also, the second term is bounded below by $2\pi/k$, so that we need only $1/c \leq 1 + 2\pi/k$, or $c \geq 1/(1 + 2\pi/k)$. The choice in equation (26) satisfies this condition, so that this c works for both cases (m even or m odd). We have proved diagonal dominance of the operator $\mathbf{A}$.

We can rewrite $\mathbf{Ac} = \tilde{\mathbf{a}}$ with $A_{mj} = \delta_j^m + \frac{16k}{\pi(m+1)}B_{jm}$ as

$$(\mathbf{I}+\mathbf{B})\mathbf{c} = \tilde{\mathbf{a}} \tag{28}$$

where $\mathbf{I}$ is the identity operator and the norm of $\mathbf{B}$, as an operator on ℓ^∞ is bounded as $|\mathbf{B}| \leq \frac{1}{1+\frac{\pi}{2k}} < 1$.

We can now find $\mathbf{c}$ using a Neumann series, as

$$\mathbf{c} = (\mathbf{I} - \mathbf{B})^{-1}\tilde{\mathbf{a}} = (\mathbf{I} + \mathbf{B} + \mathbf{B}^2 + ...)\tilde{\mathbf{a}} = \tilde{\mathbf{a}} + \mathbf{B}\tilde{\mathbf{a}} + \mathbf{B}^2\tilde{\mathbf{a}} + ...$$

Since we proved that our constant c provides diagonal dominance, we can say that $|\mathbf{B}\tilde{\mathbf{a}}|_\infty \leq c|\tilde{\mathbf{a}}|_\infty$, where $|\mathbf{x}|_\infty$ simply means the ℓ^∞ norm, i.e., that we take the largest element (after taking absolute values) of $\mathbf{x}$. We instead of using the full geometric series to compute $\mathbf{c}$ we truncate the expansion after two terms, we obtain approximation solution $\tilde{\mathbf{c}}$ given by

$$\tilde{\mathbf{c}} = (\mathbf{I} + \mathbf{B})\tilde{a}. \tag{29}$$

Then we have $|\mathbf{c} - \tilde{\mathbf{c}}|_\infty$ bounded by $c^2|\tilde{\mathbf{a}}|_\infty/(1-c)$, for

$$|\mathbf{B}^2\tilde{\mathbf{a}} + \mathbf{B}^3\tilde{\mathbf{a}}...|_\infty \leq |\mathbf{B}^2\tilde{\mathbf{a}}|_\infty + |\mathbf{B}^3\tilde{\mathbf{a}}|_\infty + \cdots \leq c^2|\tilde{\mathbf{a}}| + c^3|\tilde{\mathbf{a}}| + ... = \frac{c^2}{1-c}|\tilde{\mathbf{a}}|. \tag{30}$$

Of course this means that $|c_j - \tilde{c}_j| \leq \frac{c^2|\tilde{\mathbf{a}}|_\infty}{1-c} = \frac{4k^2|\mathbf{a}|_\infty}{\pi(2k+\pi)} = O(k^2)$ for all $j \geq 0$, where $O(k^2)$ denotes a quantity bounded by Ck^2 for some C and all k in a neighborhood of zero.

Now for the case in which $a_0 = -\pi/8$ we find that $\tilde{c}_0 = \frac{1}{2(1+\frac{8k}{3\pi})}$, so that

$$c_0 = \frac{1}{2(1+\frac{8k}{3\pi})} + O(k^2). \tag{31}$$

Since we know that $\int_\sigma [v_j]\, ds = 0$ for $j \geq 1$, while $\int_\sigma [v_0]\, ds = \pi/2$, we find that

$$\int\limits_\sigma [v] = c_0 \int\limits_\sigma [v_0] = \frac{\pi/4}{(1+\frac{8k}{3\pi})} + O(k^2).$$

Now that we have solved for c_0 when the crack lies on the interval $[0,1]$, we need to rescale the crack so that it lies on an arbitrary interval $[0,\epsilon]$. Let $v_\epsilon(x,y) = \epsilon v(\frac{x}{\epsilon}, \frac{y}{\epsilon})$. Then

$$\frac{\partial v_\epsilon}{\partial y}(x,0) = \frac{\partial v}{\partial y}(x/\epsilon, 0) \quad \text{and} \quad [v_\epsilon](x) = \epsilon[v](x/\epsilon).$$

Substituting the properties of v_ϵ into Equation (16) gives

$$\frac{\partial v_\epsilon}{\partial y}(x,0) - k[v_\epsilon](x) \quad = \quad -1,$$

which becomes

$$\frac{\partial v}{\partial y}\left(\frac{x}{\epsilon},0\right) - k\epsilon[v]\left(\frac{x}{\epsilon}\right) \quad = \quad -1$$

for $0 < x < \epsilon$. Let $\tilde{x} = \frac{x}{\epsilon}$ and $\tilde{k} = k\epsilon$. Then our last equation becomes

$$\frac{\partial v}{\partial y}(\tilde{x},0) - \tilde{k}[v]\left(\tilde{x}\right) \quad = \quad -1$$

for $0 < \tilde{x} < 1$. If we let $u = \frac{x}{\epsilon}$, then

$$\int\limits_0^\epsilon [v_\epsilon](x)dx = \epsilon \int\limits_0^\epsilon [v]\left(\frac{x}{\epsilon}\right) dx = \epsilon^2 \int\limits_0^1 [v](u)du = \frac{\frac{\pi}{4}\epsilon^2}{1+\frac{8k\epsilon}{3\pi}}.$$

Therefore our new, rescaled crack satisfies

$$\int\limits_\sigma [u]\, ds = \frac{\frac{\pi}{4}|\sigma|^2}{1+\frac{8k|\sigma|}{3\pi}} + O(|\sigma|^2 + k^2) \tag{32}$$

where the length of the crack, $|\sigma|$, simply equals ϵ and $O(|\sigma|^2 + k^2)$ means a quantity bounded by $C(|\sigma|^2 + k^2)$ for all $|\sigma|$ and k sufficiently close to zero. Solving the quadratic equation for $|\sigma|$ gives us our final length equation of

$$|\sigma| \quad = \quad \frac{\frac{8}{3\pi}Jk + \sqrt{(Jk\frac{8}{3\pi})^2 + J\pi(\mathbf{n}_\sigma \cdot \nabla u_0(\lambda))}}{\frac{\pi}{2}(\mathbf{n}_\sigma \cdot \nabla u_0(\lambda))}. \tag{33}$$

where $J = \int_\sigma [u]\, ds$. Notice that the $(\mathbf{n}_\sigma \cdot \nabla u_0(\lambda))$ term must be included because of Equation (16). In deriving equation (33) we assumed that $(\mathbf{n}_\sigma \cdot \nabla u_0)$ was a constant over the crack. Note also that although our error term gets small when k is close to zero, equation (33) works well even for values of k equal to 10 or larger, as shown in the numerical examples.

6 Conclusion

In this paper we offered a quick and efficient algorithm to find a single crack in a bounded, two-dimensional region, given that the flux over the crack is of the form $k[u]$. In order to obtain this algorithm, we relied very heavily on analysis and linear algebra to find our crack length, while our angle and midpoint results came from past research that extended very nicely to our problem. We then used a numerical optimization approach to locate small, well-separated multiple cracks. We also gave some delectable examples to illustrate our algorithm.

There are many extensions to the research we did in this paper. We think the next priority should be to reconstruct our arbitrary constant k (found in the equation $\frac{\partial u}{\partial \mathbf{n}_\sigma} = k[u]$) based on boundary measurements, rather than specifying it. We also would like an analytical algorithm to locate multiple cracks, as the method we currently use offers little mathematical insight. The method also has the major drawback of having to guess the number of cracks within our region, thereby implying that the amount of cracks is somehow known (an impractical assumption in the physical world).

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