Thermal Imaging to Recover a Defect in Two- and Three-Dimensional Objects

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Abstract

This paper focuses on the inverse problem of identifying an internal void in a bounded two or three-dimensional region. Information, in the form of heat flux and temperature, is assumed to be obtainable only on the external boundary of the region. The reciprocity gap approach with suitable test functions is used in both the two- and three-dimensional cases.

1 Introduction

Knowing the integrity of the inside of an object can be difficult without destroying the object. Thermal imaging is one method of non-destructive testing that allows one to determine if any defects exist within an object. A heat flux is applied to the external boundary of the object, and then the temperature of the object is measured on the external boundary. It is hoped that the presence of the defect yields a different temperature pattern on the surface. The flow of heat through the interior of the object is assumed to be governed by the heat equation

Once the presence of a defect has been detected, the problem becomes one of locating the defect and determining its size. This type of problem can be considered with or without time dependence. In this paper, I will consider the two- and three-dimensional cases without time dependence. The type of defect examined is a circular/spherical hole or "void" where no material exists. These methods are developed using past research by Bryan, Krieger, and Trainor [1] and Spring and Talbott [2] as a basis.

1.1 Forward Problem

The steady-state forward problem is assumed to be governed by the time-independent heat equation, or Laplace's Equation. Let Ω be a bounded region in $\mathbb{R}^n$ and $\partial\Omega$ denote the $(n-1)$-dimensional boundary of Ω. A void D may exist inside Ω where no material is present; the boundary of the void is denoted ∂D. The temperature of Ω at position $(x_1, x_2, \ldots, x_n)$ is denoted by $u(x_1, x_2, \ldots, x_n)$, where u obeys Laplace's Equation:

$$\Delta u := \sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = 0. \tag{1}$$

A heat flux $g(x_1, x_2, \ldots, x_n)$ is applied to $\partial\Omega$. In the steady state case the net heat flux on $\partial\Omega$ must be zero, that is, $\int_{\partial\Omega} g\, ds = 0$, since energy can neither be created nor destroyed. No heat flows across ∂D since we are assuming the boundary of the void is insulating. These boundary conditions can be quantified as

$$\frac{\partial u}{\partial \mathbf{n}} := \nabla u \cdot \mathbf{n} = g \quad \text{on} \quad \partial\Omega \tag{2}$$

$$\frac{\partial u}{\partial \mathbf{n}} = 0 \quad \text{on} \quad \partial D \tag{3}$$

with $\mathbf{n}$ the outward unit normal vector on $\partial\Omega$ or ∂D as appropriate. In the future we will in fact take $\mathbf{n}$ on ∂D to point OUT of D (away from the center when D is a disk or sphere).

Equations (1)-(3) possess a unique solution if we add the additional normalization

$$\int_{\partial\Omega} u\, dx = 0.$$

2 Two-Dimensional Inverse Problem

2.1 Reciprocity Gap Approach

For the two-dimensional inverse problem, we need to find the center (a, b) and radius R of a circular void D contained within Ω. One method to solve the inverse problem is the Reciprocity Gap Approach. The Reciprocity Gap formula is derived from the Divergence Theorem and Green's Second Identity. Let v be a harmonic "test function" on Ω. The Reciprocity Gap formula is

$$\int_{\partial\Omega} \left(u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}} \right) ds = \int_{\partial D} u\frac{\partial v}{\partial n} ds \tag{4}$$

This is easily obtained by applying Green's second identity to u and v on the region $\Omega \setminus D$ (v is harmonic on Ω, hence on $\Omega \setminus D$). Note we assume that $\mathbf{n}$ on ∂D points out of D (into $\Omega \setminus D$).

The left hand side of equation (4) can be calculated using known data. A chosen heat flux $g(x, y) = \frac{\partial u}{\partial \mathbf{n}}$ is applied and the temperature $u(x, y)$ is measured on $\partial\Omega$. Let us define

$$RG(v) := \int_{\partial\Omega} u\frac{\partial v}{\partial \mathbf{n}} - v\frac{\partial u}{\partial \mathbf{n}} ds \tag{5}$$

so that equation (4) can be written

$$RG(v) = \int_{\partial D} u\frac{\partial v}{\partial n} ds \tag{6}$$

Our goal is to use cleverly chosen functions v and our ability to compute $RG(v)$ in (6) to extract information about D.

2.2 Test Functions

Different classes of test functions can be used to locate a void D and determine its size. One class of harmonic test functions is

$$v(x, y) = \ln\left((x - p)^2 + (y - q)^2\right) \tag{7}$$

with (p, q) a point outside $\bar{\Omega}$, the closure of Ω.

Let us assume that D is a disk centered at (a, b), of radius R. We can parameterize ∂D as $x = a+R\cos(\theta)$, $y = b+R\sin(\theta)$ for $0 \leq \theta < 2\pi$. The unit normal vector which points AWAY from (a, b) is given by $\mathbf{n} = \langle \cos\theta, \sin\theta \rangle$ on ∂D. The Reciprocity Gap Formula (6) with test function v defined by (7) can now be written, after a bit of algebra, as

$$RG(v) = \int_0^{2\pi} u(R, \theta)\frac{2(a - p)\cos\theta + 2(b - q)\sin\theta + R}{(a + R\cos\theta - p)^2 + (b + R\sin\theta - q)^2} R\, d\theta \tag{8}$$

where we use $ds = R\,d\theta$.

Let us assume that we have a small radius defect. We can approximate (8) by using the first term of the Taylor approximation about $R = 0$ which yields

$$RG(v) = \int_0^{2\pi} u(R,\theta)\frac{2(a-p)\cos\theta + 2(b-q)\sin\theta}{(a-p)^2 + (b-q)^2}R\ d\theta \tag{9}$$

We will use equation (9) to locate the center of the void.

2.3 Locating the Center of the Void

To find the center of the void, (9) can be written in the form

$$RG(v) = \frac{2\mathbf{r}\cdot\mathbf{J}}{|\mathbf{r}|^2} \tag{10}$$

where

$$\begin{aligned} \mathbf{J} &= \langle J_c, J_s\rangle \\ \mathbf{r} &= \langle a-p, b-q\rangle \\ J_c &= \int_0^{2\pi} u(R,\theta)\,(\cos\theta)\,R\ d\theta \\ J_s &= \int_0^{2\pi} u(R,\theta)\,(\sin\theta)\,R\ d\theta \end{aligned}$$

To simplify further, equation (10) can be written as

$$RG(v) = \frac{2\,|\mathbf{J}|\cos\alpha}{|\mathbf{r}|} \tag{11}$$

with α the angle between the vectors $\mathbf{J}$ and $\mathbf{r}$. To recover the center (a,b), we know $\mathbf{J}$ will remain constant since it depends on the radius R and $u(R,\theta)$, but not (p,q). The vector $\mathbf{r}$ will vary as the distance between the chosen point (p,q) and the center (a,b) varies. Figure 1 illustrates the relationship between $\mathbf{J}$ and $\mathbf{r}$. Since $RG(v)$ can be calculated at any (p,q) of our choosing outside Ω, we choose many points (p,q) that form a circle slightly outside Ω. It's easy to see from equation (11) that at exactly two points around the circle we have $RG(v) = 0$; these points will occur when $\alpha = \frac{\pi}{2}, \frac{3\pi}{2}$. A line segment, l, connecting the two points will pass through the center of D.

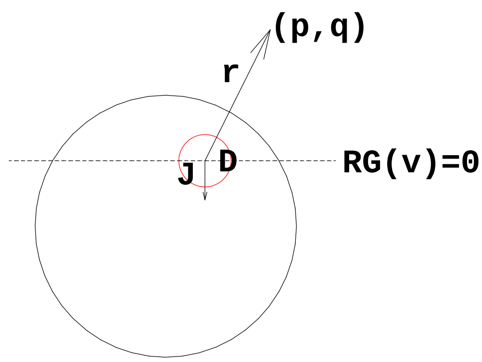

Figure 1: Relationship of **J** and **r**

Strategically chosen points (p, q) can now reveal the exact center of D. Starting at one end of l (in Ω), assume a given point P on l is the center. Choose a point j a distance d from the assumed center P such that the line connecting it to P is orthogonal to l. Next, choose a point k the same distance d from P, but so that the line connecting it to the assumed center forms an angle β with l. If the assumed center P is the true center, equation (11) shows that the following relationship will hold

$$RG(j) = \frac{RG(k)}{\cos \beta} \tag{12}$$

since $\mathbf{J}$ is orthogonal to l. By choosing a sequence of such assumed centers at regular intervals along l we may approximately locate the true center (at that point where (12) holds). A plot of the relationship (12) for all the assumed centers will easily show the location of the true center. Figure 2 displays the method for determining the center using two points j, k. A computational example is given below. It remains to be shown that this approach recovers the center uniquely.

Note that we also recover the value of the vector $\mathbf{J}$.

2.4 Determining the Radius R

Once the center is known, the radius can be determined. Let $J = J_c + J_s$. Polar coordinates are easiest to use with this section so let use define ρ as the distance from the center (a, b) of the defect. In this case we have that $\frac{\partial u}{\partial \mathbf{n}}$ simply becomes $\frac{\partial u}{\partial \rho}$ on ∂D. We will write $u(\rho, \theta)$ to denote u in the polar coordinate system centered at (a, b).

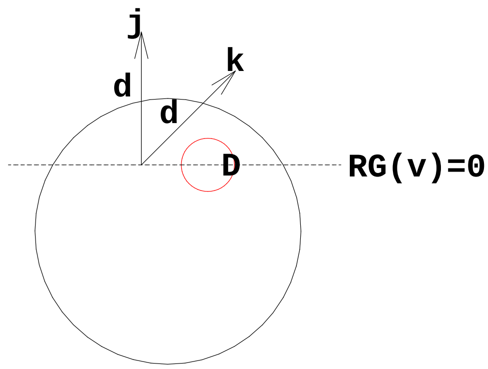

Figure 2: Method to find Center

We need to approximate $u(R, \theta)$ (that is, u on ∂D) to determine R. Let's consider $u_0(x, y)$ to be the solution to equations (1)-(2) when no defect is present. If a defect is present, $u(x, y)$ should be close to u_0, that is, $u(x, y) = u_0(x, y) + w(x, y)$ with $w(x, y)$ a "small" adjustment function. Then $w = u - u_0$ is harmonic on $\Omega \setminus D$. Also, on $\partial\Omega$ we have $\frac{\partial w}{\partial \mathbf{n}} = 0$ since u and u_0 have the same normal derivatives. On ∂D we have

$$\frac{\partial w}{\partial \rho} = -\frac{\partial u_0}{\partial \rho} \tag{13}$$

We will approximate w by assuming that $\frac{\partial w}{\partial \mathbf{n}}$ is negligible on $\partial\Omega$, of order $O(R^3)$, while the other approximations we make will be order $O(R^2)$. A proof of this fact can be given using the techniques in [1].

Accordingly, let $\tilde{w}$ denote the harmonic function on $\mathbb{R}^2 \setminus D$ which satisfies $\frac{\partial \tilde{w}}{\partial \rho} = -\frac{\partial u_0}{\partial \rho}$ on ∂D. Using the same assumption that R is small, we can approximate $\nabla u_0(x, y)$ on ∂D to be $\nabla u_0(a, b)$. This means

$$\frac{\partial u_0}{\partial \rho} = \nabla u_0(a, b) \cdot \langle \cos\theta, \sin\theta \rangle$$

up to an error of $O(R)$. We then can say

$$\frac{\partial \widetilde{w}}{\partial \rho} = -\nabla u_0(a, b) \cdot \langle \cos\theta, \sin\theta \rangle$$

on ∂D. We can find a harmonic $\widetilde{w}$ that satisfies (2.4); it is

$$\widetilde{w}(\rho, \theta) = R^2 \frac{\nabla u_0(a, b) \cdot \langle \cos\theta, \sin\theta \rangle}{\rho}$$

On ∂D, $\rho = R$ we have

$$\widetilde{w}(R,\theta) = R\nabla u_0(a,b) \cdot \langle \cos\theta, \sin\theta \rangle \,.$$

It can be shown (again, using the techniques of [1]) that $\widetilde{w}(R,\theta) = w(R,\theta) + O(R^2)$.

We next approximate $u = u_0 + w \approx u_0 + \widetilde{w}$, which yields

$$\widetilde{u}(R,\theta) = u_0(R,\theta) + R\nabla u_0(a,b) \cdot \langle \cos\theta, \sin\theta \rangle$$

on ∂D. For a final approximation, if we use the tangent plane approximation to approximate u_0 at (a,b), we have

$$u_0(R,\theta) = u_0(a,b) + \nabla u_0(a,b) \cdot \langle R\cos\theta, R\sin\theta \rangle + O(R^2)$$

Putting all of this into (2.4), we now have

$$\widetilde{u}(R,\theta) = u_0(a,b) + 2R\nabla u_0(a,b) \cdot \langle \cos\theta, \sin\theta \rangle \tag{14}$$

We now substitute (14) into (6) and integrate over ∂D (i.e., $0 \leq \theta < 2\pi$); the $u_0(a,b)$ term falls out when integrated, and we end with the following relationship

$$J = J_c + J_s \approx 2\pi R^2 \left(\frac{\partial u_0}{\partial x}(a,b) + \frac{\partial u_0}{\partial y}(a,b) \right) \tag{15}$$

to order $O(R^3)$.

We have already recovered the center (a,b) and the vector $\mathbf{J}$, in particular, the values of J_s and J_c. Since u_0 is known, we can solve equation (15) for R.

2.5 Example

Let's consider an example of a two-dimensional steady state inverse problem. Let Ω be the unit disk in $\mathbb{R}^2$. Let $g(\theta) = \sin\theta$, which means heat flows in the top and out the bottom of Ω. Let D be a circular void centered at $(0.3, -0.2)$ with radius 0.15. The temperature $u(x,y)$ on $\partial\Omega$ is calculated and plotted using Femlab. On the following plot, the green curve shows the temperature with no defect, and the red curve shows the temperature with the defect. A difference can be seen between the temperature with no defect present and the temperature with a defect present. In this it's easy to see

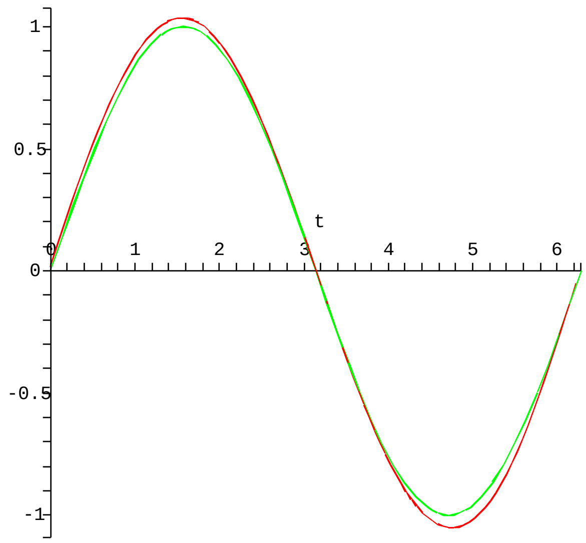

Figure 3: Plot of Temperature Differences

that $u_0(x, y) = y$.

Next, the $RG(v)$ values are calculated and plotted around a circle with radius 1.1 centered at $(0, 0)$. Figure 4 shows $RG(v) = 0$ at two points; when $\theta \approx 3.324, 6.101$. Figure 5 shows when these points are connected, the line l is equivalent to $y \approx -0.199$, which passes through the center. The $RG(v)$ values are calculated at points j, k with distance $d = 2$. Point k was chosen with $\alpha = \pi/4$. Figure 6 shows when plotted, it is easily seen (12) holds at one point, $x \approx 0.310$. The center of D is now known, $(a, b) \approx (0.310, -0.199)$. These values are now used to calculate the radius R. When ∇u_0 is calculated at $(0.310, -0.199)$, the resulting vector is $\langle 0, 1\rangle$. J is calculated to be $J \approx 0.146$. The relationship (15) shows $R \approx -0.152, 0.152$. Since the radius cannot be less than zero, we know $R \approx 0.152$.

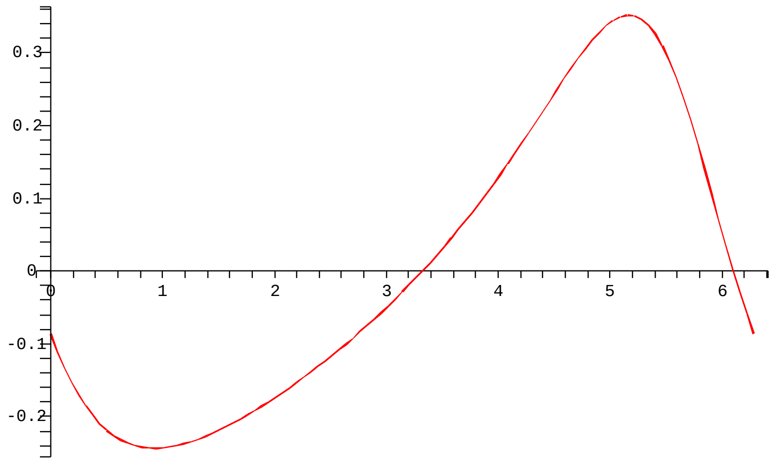

Figure 4: RG(v) values

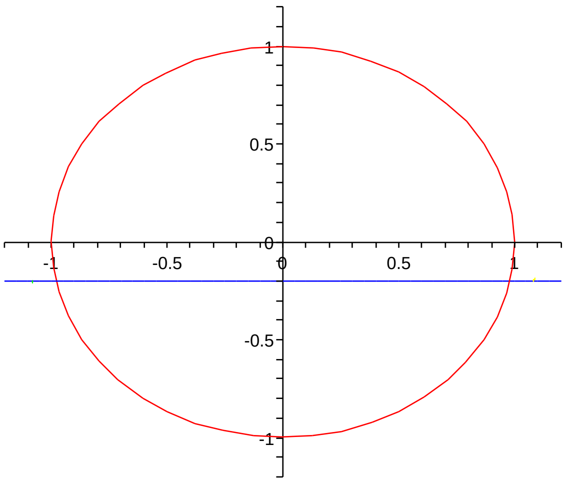

Figure 5: RG(v)=0

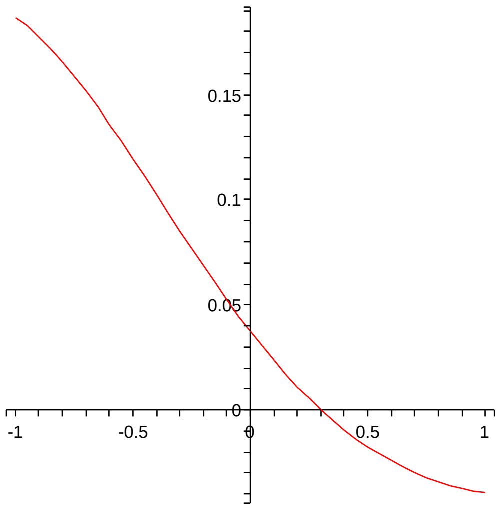

Figure 6: Center computation

3 Three-Dimensional Inverse Problem

3.1 Test Functions

The three-dimensional inverse problem is to find the center (a, b, c) and radius R of a single void D. Note that the Reciprocity Gap approach extends to the three-dimensional case, for equation (4) still holds. The recovery procedure is very similar to the two-dimensional case.

We will again employ the reciprocity gap functional, but with harmonic test functions of the form

$$v(x, y, z) = \frac{1}{\sqrt{(x-p)^2 + (y-q)^2 + (z-s)^2}} \tag{16}$$

with (p, q, s) a point outside Ω. We parameterize the surface of the void D as

$$\begin{aligned} x &= a + R\sin\phi\cos\theta \\ y &= b + R\sin\phi\sin\theta \\ z &= c + R\cos\phi \end{aligned}$$

where $0 \leq \theta < 2\pi, 0 \leq \phi \leq \pi$. Note that surface measure is given by $dS = R^2 \sin\phi \; d\theta$. With this parameterization the (outward) unit normal vector on ∂D is

$$\mathbf{n} = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle \tag{17}$$

The Reciprocity Gap Formula (6) can now be written as

$$RG(v) = \int_0^{2\pi} \int_0^{\pi} u(R,\theta,\phi) \frac{\partial v}{\partial \mathbf{n}} R^2 \sin\phi \; d\phi \; d\theta \tag{18}$$

with

$$\frac{\partial v}{\partial \mathbf{n}} = \frac{(p-a)\sin\phi\cos\theta + (q-b)\sin\phi\sin\theta + (s-c)\cos\phi - R}{[(a+R\sin\phi\cos\theta - p)^2 + (b+R\sin\phi\sin\theta - q)^2 + (c+R\cos\phi - s)^2]^{3/2}} \tag{19}$$

We assume that R is close to zero, so that we can approximate (19) by using the first term of the Taylor approximation about $R = 0$ which yields

$$\frac{\partial v}{\partial \mathbf{n}} \approx \frac{(p-a)\sin\phi\cos\theta + (q-b)\sin\phi\sin\theta + (s-c)\cos\phi}{[(a-p)^2 + (b-q)^2 + (c-s)^2]^{3/2}} \tag{20}$$

We now show how to use this to approximate the center of the void.

3.2 Locating the Center of the Void

To find the center of the void, note that (18) (with the approximation (20)) can be written as a dot product

$$RG(v) = \frac{\mathbf{r} \cdot \mathbf{J}}{|\mathbf{r}|^3} \tag{21}$$

where

$$\begin{aligned}
\mathbf{r} &= \langle p-a, q-b, s-c \rangle \\
\mathbf{J} &= \langle J_{sc}, J_{ss}, J_c \rangle \\
J_{sc} &= \int_0^{2\pi} \int_0^{\pi} u(R,\theta,\phi) \sin\phi\cos\theta R^2 \sin\phi \; d\phi d\theta \\
J_{ss} &= \int_0^{2\pi} \int_0^{\pi} u(R,\theta,\phi) \sin\phi\sin\theta R^2 \sin\phi \; d\phi d\theta \\
J_c &= \int_0^{2\pi} \int_0^{\pi} u(R,\theta,\phi) \cos\phi R^2 \sin\phi \; d\phi d\theta
\end{aligned}$$

To simplify, (21) can be expanded to

$$RG(v) = \frac{|\mathbf{J}| \cos \alpha}{|\mathbf{r}|^2} \tag{22}$$

with α the angle between $\mathbf{J}$ and $\mathbf{r}$.

In recovering the center (a, b, c), we know $\mathbf{J}$ will remain constant since it depends on the radius R and $u(R, \theta, \phi)$, neither of which vary for a single input flux. The vector $\mathbf{r}$ will vary as the distance between the chosen point (p, q, s) and the center (a, b, c) varies. Since $RG(v)$ can be calculated at any (p, q, s) outside Ω of our choosing, we choose points (p, q, s) that form a sphere larger than Ω. Then it is easy to see from equation (22) that $RG(v) = 0$ exactly when $\mathbf{J}$ and $\mathbf{r}$ are orthogonal; these points will form a plane P that passes through the center (a, b, c) of D. Thus we can easily identify this plane P.

Strategically chosen points (p, q, s) can now reveal the exact center of D. Assume that a point c^* on the plane P (within Ω) is the center of D. Choose a point j a distance d from c^* such that the line L_j connecting j to c^* is orthogonal to P. Next, choose a point k at the same distance d from c^* with L_k at an angle β to L_j. Choose a third point m a distance d from c^* with L_m also at an angle β to L_j. The points j, k, and m should be distinct from one another. If c^* is the true center, then equation (22) shows that following relationship will hold

$$RG(j) = \frac{RG(k)}{\cos \beta} = \frac{RG(m)}{\cos \beta} \tag{23}$$

since $\mathbf{J}$ is orthogonal to P.

By testing equation (23) for "each" point c^* in the section of P intersecting Ω, we can identify the true center. A plot of (23) for all c^* will easily identify the center of D. Figure 7 illustrates the relationship between the points j, k, m and how to find the center. As in the two-dimensional case, it remains to be shown that this approach recovers the center uniquely.

3.3 Determining the Radius R

A theorem for determining the radius of a void D is

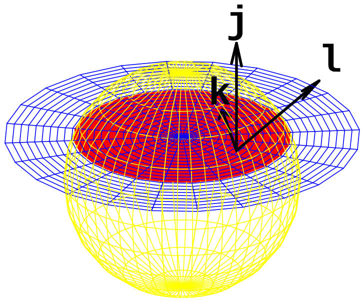

Figure 7: Method to Find the Center

Theorem: *For any harmonic test function* $v(x,y,z) = \frac{1}{\sqrt{(x-p)^2+(y-q)^2+(z-s)^2}}$ *with* (p,q,s) *outside of* Ω *we have*

$$RG(v) = 2\pi R^3 \left(\nabla u_0(a,b,c)\cdot \nabla v(a,b,c)\right) + O(R^4) \tag{24}$$

where (a,b,c) *is the center of a void of radius* R *and* u_0 *is the harmonic function on* Ω *with Neumann data* g.

The following is the development of this theorem. To determine the radius R, spherical coordinates are easiest to use. Let's start by considering the temperature $u(x,y,z)$ as the sum of $u_0(x,y,z)$ (the temperature with no defect present) and a "small" adjustment function $w(x,y,z)$. The function u_0 is harmonic on Ω and w is harmonic on $\Omega\backslash D$. Moreover, $\frac{\partial u_0}{\partial \mathbf{n}} = g$ on $\partial\Omega$; as such, the function u_0 is known or computable.

For w, we note that $w = u - u_0$ and so

$$\frac{\partial w}{\partial \mathbf{n}} = 0 \quad \text{on } \partial\Omega \tag{25}$$

$$\frac{\partial w}{\partial \mathbf{n}} = -\frac{\partial u_0}{\partial \mathbf{n}} \quad \text{on } \partial D \tag{26}$$

We will "ignore" the boundary condition (25), and instead define w as that function which is harmonic on $\mathbb{R}^3$ which satisfies the boundary condition (26) and decays to zero at infinity. Using the techniques in [1], one can show that this introduces error of order $O(R^4)$, which will not affect our theorem.

On ∂D, (26) can be written as $-\nabla u_0(x,y,z)\cdot\mathbf{n}$, which is approximately $-\nabla u_0(a,b,c)\cdot\mathbf{n} + O(R)$ assuming D is small. If we consider $\widetilde{w}$ to be an approximation of w, the solution to $\Delta\widetilde{w} = 0$ in $\mathbb{R}^3\backslash D$ with $\frac{\partial \widetilde{w}}{\partial \mathbf{n}} = -\nabla u_0(a,b,c)\cdot\mathbf{n}$ is

$$\widetilde{w}(\rho,\theta,\phi) = \frac{R^3 \nabla u_0(a,b,c)\cdot\mathbf{n}}{2\rho^2} \tag{27}$$

with ρ as the radial component in spherical coordinates centered at (a, b, c). Again, with the techniques of [1] we can show that $w = \widetilde{w} + O(R^4)$.

On ∂D, ρ is the radius R, which makes (27)

$$\widetilde{w}(R, \theta, \phi) = \frac{1}{2} R \nabla u_0(a, b, c) \cdot \mathbf{n}. \tag{28}$$

Next, we need to approximate $u_0(x, y, z)$ on ∂D. Doing so gives us,

$$u_0(x, y, z) = u_0(a, b, c) + R\nabla u_0(a, b, c) \cdot \mathbf{n} + O(R^2). \tag{29}$$

We can now write an approximation of $u(x, y, z)$ on ∂D using (28) and (29) which yields

$$u = u_0 + w \approx u_0(a, b, c) + \frac{3}{2} R\nabla u_0(a, b, c) \cdot \mathbf{n} \tag{30}$$

A similar approximation for $\frac{\partial v}{\partial \mathbf{n}}$ on ∂D yields

$$\frac{\partial v}{\partial \mathbf{n}} = \nabla v(x, y, z) \cdot \mathbf{n} \approx \nabla v(a, b, c) \cdot \mathbf{n} + O(R). \tag{31}$$

Thus, substituting (30) and (31) into (6), we have to order $O(R^4)$,

$$\int_{\partial D} u \frac{\partial v}{\partial \mathbf{n}}\, dA = \int_0^{2\pi} \int_0^{\pi} \left(u_0(a, b, c) + \frac{3}{2} R\nabla u_0(a, b, c) \cdot \mathbf{n} \right) (\nabla v(a, b, c) \cdot \mathbf{n})\, R^2 \sin\phi d\phi d\theta \tag{32}$$

The $u_0(a, b, c)$ term will fall out when integrated. The result is

$$RG(v) = 2\pi R^3 \left(\nabla u_0(a, b, c) \cdot \nabla v(a, b, c)\right) \tag{33}$$

up to an error of $O(R^4)$. This completes the proof of the theorem.

We can now locate a void D and determine its size in a three-dimensional object.

3.4 Example

Let Ω be the unit sphere in $\mathbb{R}^3$. Let $g(\phi) = \cos\phi = z$ with the usual spherical coordinates parameterization, which means that heat flows in the top and out the bottom of Ω. For this heat flux, $u_0(x, y, z) = z$. Let D be a spherical void centered at $(0.2, 0.1, 0.5)$ with radius 0.2. The temperature $u(x, y, z)$ is calculated from "measured" data (computed using Femlab, at total of 1300 on the surface of Ω, on a 50 point longitudinal by 26 point latitudinal grid).

In Figure 8 the $RG(v)$ values are calculated and plotted around a sphere with radius 2. The plot shows the ϕ (latitude) axis, labelled by point numbers 1 to 26, and the θ axis, labelled by point numbers 1 to 50. The value of $RG(v)$ at the relevant point is indicated by elevation and color. Figure 8 shows that the set $RG(v) = 0$ forms a plane through Ω at $\phi \approx 1.508$. The plane P is approximately $z = 0.4363$, which approximately passes through the center of D. The $RG(v)$ values are calculated at points j, k, m with distance $d = 3$. Points k and m are chosen with $\beta = \pi/4$. Figure 9 shows (23) holds at one point, $(x, y) \approx (0.2421, 0.0622)$. The center of D is now known, $(0.2421, 0.0622, 0.4363)$. These values are now used to calculate the radius R. When ∇u_0 is calculated at $(0.2421, 0.0622, 0.4363)$, the resulting vector is $\langle 0, 0, 1\rangle$. To evaluate for R, we take $(p, q, s) = (1.2, 1.2, 1.2)$. The radius is calculated to be $R = 0.1725$ using (33). Figures 10 and 11 show the position of the recovered void relative to Ω and D.

4 Conclusion

The above method shows a circular/spherical void can be located and its size determined from the boundary data in both two- and three-dimensional objects. A few ways to continue this research would be to consider multiple defects, multiple heat influxes, time dependence, cracks, and other types of inclusions. It may also be possible to develop more "methodical" approaches to the computations, and optimize them for more stable reconstructions.

References

[1] Bryan, K., Rachel Krieger, and Nic Trainor, *Imaging of multiple linear cracks using impedance data*, to appear in J. of Computational and Applied Math.

[2] Spring, H., Shannon Talbott, *Thermal imaging of circular inclusions within a two-dimensional region*, Rose-Hulman Mathematics Technical Report 05-02, August 2005.

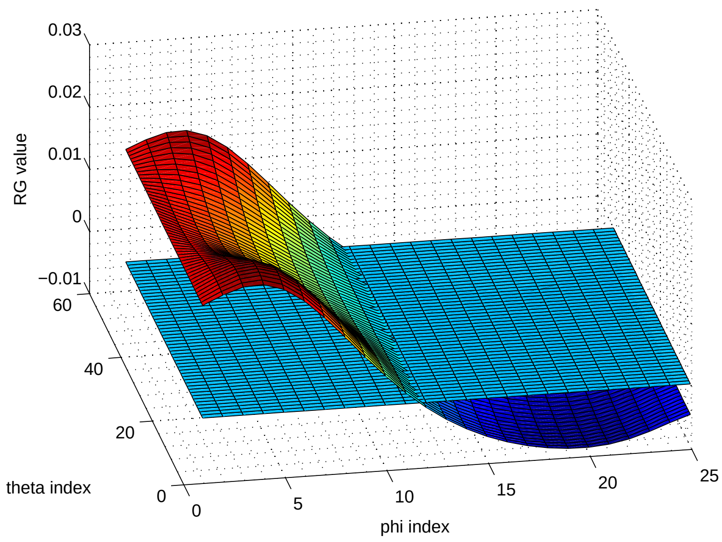

Figure 8: RG(v)

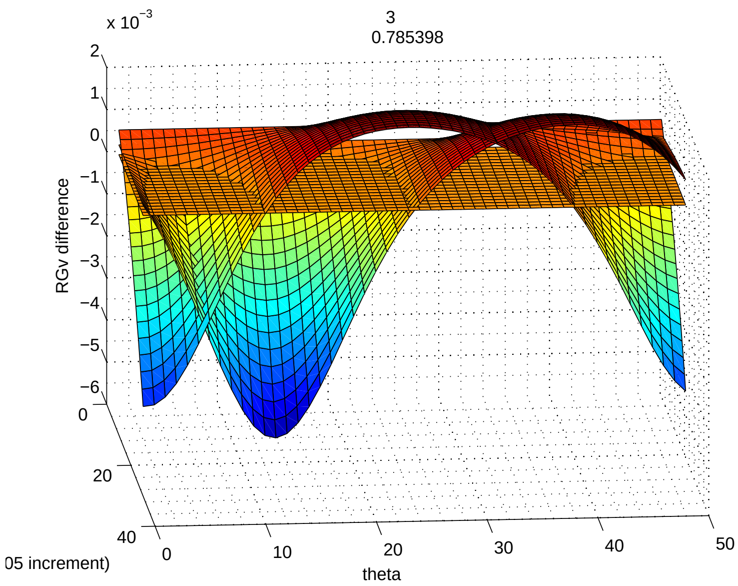

Figure 9: Center of D

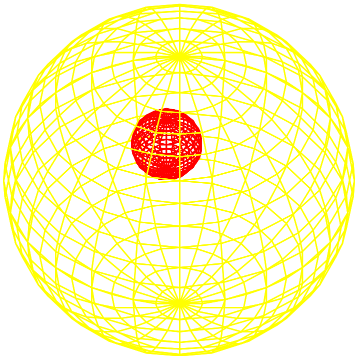

Figure 10: Ω with Real void D

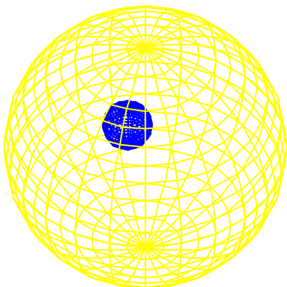

Figure 11: Ω with Recovered void D