

# General Technical Remarks on PDE's and Boundary Conditions

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## 1 Introduction

You may have noticed that when we analyzed the heat equation on a bar of length 1 and I talked about the equation  $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$  holding on the bar, I always used the strict inequalities  $0 < x < 1$ , NOT  $0 \leq x \leq 1$ . I also stated that the equation should hold for  $t > 0$ , not  $t \geq 0$ . I made similar remarks for the wave equation and Laplace's equation, and used strict inequalities to define where the PDE should hold, rather than weak inequalities.

This is generally true: when analyzing a PDE on a region  $D$ , we don't generally talk about the PDE holding true at the boundary of the region—in fact, this is usually a meaningless statement, for reasons I'll explain below.

You may also have noticed that when we require a solution to a PDE to assume certain boundary or initial values, we don't always get quite what we want. For example, in solving Laplace's equation with Dirichlet boundary data on the unit disk we didn't really obtain  $u(1, \theta) = h(\theta)$ , at least not in the pointwise sense. Rather, all we could generally assert is that  $\lim_{r \rightarrow 1^-} \|u(1, \cdot) - h\|_2 = 0$ . This last statement does not force  $\lim_{r \rightarrow 1^-} u(1, \theta) = h(\theta)$  for any particular  $\theta$ !

These are somewhat subtle but important issues that I've put off until now, when we have a number of examples that illustrate what's going on.

## 2 Domains and Derivatives

### 2.1 Open and Closed Sets

Given a point  $x$  in  $\mathbb{R}^n$  we use the notation  $B_r(x)$  to denote the set of points  $y$  such that  $|x - y| < r$ , where “ $|x - y|$ ” is the usual Pythagorean distance  $(\sum_j (x_j - y_j)^2)^{1/2}$ . The set  $B_r(x)$  is called an *open ball* of radius  $r$  around  $x$ . Obviously  $B_{r_1}(x) \subset B_{r_2}(x)$  if  $r_1 < r_2$ .

A subset  $D$  of  $\mathbb{R}^n$  is called *open* if, given any point  $x \in D$ , there is some  $r > 0$  such that  $B_r(x) \subset D$ . The open ball  $B_r(x)$  in  $\mathbb{R}$  is just the interval

$(x - r, x + r)$ . In fact, any interval  $(a, b)$  in the reals is open by the above definition (check this!)

Although we don't need this definition, a subset  $D$  of  $\mathbb{R}^n$  is called *closed* if  $\mathbb{R}^n \setminus D$  is open. But a set can be neither open nor closed, e.g.,  $[a, b]$ .

There are various ways to define the boundary  $\partial D$  of a set  $D$ . Here's one: We'll say that  $x \in \partial D$  if EVERY ball  $B_r(x)$  contains points in  $D$  and points in  $\mathbb{R}^n \setminus D$ . The *closure* of a set  $D$ , written  $\bar{D}$ , is the set  $D \cup (\partial D)$ . It's worth noting that if  $x \in \partial D$  then we can pick a sequence of points in  $D$  that converge to  $x$ . Try proving this from the definition of  $\partial D$ !

A set  $D \subset \mathbb{R}^n$  is *connected* if any two points in  $D$  can be joined together by a continuous curve which doesn't leave  $D$ —intuitively,  $D$  consists of a single piece or “component.”

Finally, a *domain*  $D$  in  $\mathbb{R}^n$  is an open connected set. The set  $D$  need not be bounded. However, if  $D$  is in fact unbounded people will often refer to it as an “unbounded domain.”

## 2.2 Derivatives in $\mathbb{R}$

I've used the notation  $C^k$  somewhat informally so far, to mean that a function has at least  $k$  derivatives, all continuous. Let me make this a bit more precise, especially in regard to open and closed sets.

Just as a reminder, if  $f$  is a function defined on some open ball  $B_r(x_0)$  in  $\mathbb{R}^n$  we say that  $f$  is continuous at  $x_0$  if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0).$$

Let's start in just  $\mathbb{R}$ . Let  $I$  denote the open interval  $(a, b)$ . We say that a function  $f \in C^k(I)$  if  $f$  is defined at all points in  $I$ ,  $f$  is  $k$  times differentiable at each point in  $I$ , and the derivatives are all continuous functions on  $I$ . The case  $k = 0$  is used to indicate simply that a function is continuous, and  $k = \infty$  means the function is infinitely differentiable, that is,  $f \in C^k(I)$  for all  $k \geq 0$ .

As an example, let  $I = (0, 1)$ . The function  $f(x) = 1/x$  is in  $C^k(I)$  for all  $k$ , that is,  $f \in C^\infty(I)$ . The vertical asymptote at  $x = 0$  isn't a problem, because  $x = 0$  isn't in  $I$ . Note I can also legitimately write  $f \in C^{37}(I)$ , because  $f$  indeed has 37 continuous derivatives on  $I$ . We can say a function is in  $C^k(I)$  even if it has more than  $k$  continuous derivatives—it just has to have at least  $k$ .

Now let  $I = [a, b]$ , a closed interval, and  $f$  a function defined on  $I$  (including the endpoints). We say that  $f$  is continuous on  $I$ , and write  $f \in C^0(I)$ , if  $f \in C^0(a, b)$  ( $f$  is continuous on the open interval) and if

$$f(a) = \lim_{x \rightarrow a^+} f(x), \quad f(b) = \lim_{x \rightarrow b^-} f(x). \quad (1)$$

However, suppose that  $f$  is some function defined only on an open interval  $I = (a, b)$  but not at the endpoints, and  $f \in C^0(a, b)$ . It may be possible to extend  $f$  to a continuous function on the closed interval  $\bar{I} = [a, b]$ . Specifically, if both one-sided limits in equation (1) exist we can take them as the *definition* of  $f(a)$  and  $f(b)$ . In this case  $f$  extends to a function  $C^0(\bar{I})$ . Technically since the domain of  $f$  has changed we should rename the function, but no one ever does.

Here's some examples. Let  $I = (0, 1)$  and let  $f$  be defined as  $f(x) = x \ln(x)$  on  $I$ . This function IS continuous on  $I$ , but it can actually be extended to a continuous function on  $\bar{I} = [0, 1]$ , by setting  $f(0) = 0$  and  $f(1) = 0$ . Just graph  $f$  or compute the limits! But note that the original definition of  $f$ , which involves the log, doesn't allow us to just plug in  $x = 0$ .

Here's another example: let  $I = (0, 1)$  again, but take  $f(x) = 1/x$ . The function  $f \in C^0(I)$ , but we can't extend  $f$  to be continuous on the closed interval, since  $\lim_{x \rightarrow 0^+} 1/x$  doesn't exist. We could extend to  $x = 1$ , though.

Given a closed interval  $I = [a, b]$  in  $\mathbb{R}$  we'll say that  $f \in C^k([a, b])$  if  $f \in C^k(a, b)$  and every derivative  $f^{(j)}$  for  $j \leq k$  can be extended to be continuous on the closed interval  $[a, b]$ .

### 2.3 Derivatives in $\mathbb{R}^n$

I defined above what it means for a function  $f$  of  $n$  variables to be continuous. Given an open set  $D$  in  $\mathbb{R}^n$  we'll say that  $f \in C^0(D)$  if  $f$  is continuous at each point  $x \in D$ . Of course this assumes that  $f$  is defined at each point in  $D$ .

Suppose that  $f$  is defined on  $\bar{D}$ , the closure of  $D$ . We'll say that  $f \in C^0(\bar{D})$  if  $f \in C^0(D)$  and if

$$\lim_{x \rightarrow x_0} f(x) = f(x_0)$$

for each point  $x_0 \in \partial D$ , where  $x$  approaches  $x_0$  from inside  $D$  (that is,  $x$  must stay in  $D$  as it approaches  $x_0$ ). As in  $\mathbb{R}$ , it may be the case that a function

$f$  defined on some open set  $D$  can be extended to a continuous function on  $\bar{D}$ .

Let  $D$  be an OPEN set in  $\mathbb{R}^n$  (so around any point  $x$  in  $D$  we can put a small ball  $B_r(x)$  that is contained in  $D$ ). Let  $f$  be a function defined on  $D$ . We'll say that  $f \in C^m(D)$  if all partial derivatives

$$\frac{\partial^k f}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}}$$

where  $k_1 + \cdots + k_n = k$  exist and are continuous, for each  $k \leq m$ . We'll say that  $f \in C^k(\bar{D})$  if each of these partials can be extended to a continuous function on  $\bar{D}$ .

### 3 Back to PDE's

#### 3.1 Why Open Sets?

Consider the following boundary value problem: Find a function  $u$  which satisfies

$$u''(x) = \frac{1}{x}$$

for  $0 < x < 1$  (note the strict inequalities) with the boundary conditions  $u(0) = 0$  and  $u(1) = 0$ . These last two boundary conditions are to be interpreted in the sense that we want

$$\lim_{x \rightarrow 0^+} u(x) = 0, \quad \lim_{x \rightarrow 1^-} u(x) = 0.$$

You can check that the unique solution is given by  $u(x) = x \ln(x)$ .

But what if I'd worked on the closed interval  $[0, 1]$ ? The first problem is that  $1/x$  isn't defined on this interval, and can't be extended, so  $u''(x) = 1/x$  is meaningless on this interval. But even if you ignore that problem, the solution we found is not twice-differentiable on  $[0, 1]$  (more precisely, can't be extended to  $C^2([0, 1])$ ). If we'd tried to work on  $[0, 1]$  we'd have to conclude that there is no solution.

This is a typical example of why we don't ask for a solution to a PDE (or even an ODE) to actually satisfy the PDE on the boundary of the region. If, for example, the PDE is second order, then asking for the solution to satisfy the PDE on the boundary is to require the solution to have second derivatives

that extend continuously onto the boundary of the region. This is asking too much—it won’t likely happen. And the boundary or initial conditions will only involve the function or its first derivatives anyway, so why ask for two derivatives when only one (or none) are needed?

As another example, consider solving Laplace’s equation on the OPEN unit disk  $D$  in  $\mathbb{R}^2$ , with Dirichlet boundary data  $h(\theta) = \theta(2\pi - \theta)$  (where the boundary is parameterized in the usual way,  $x_1 = \cos(\theta)$ ,  $x_2 = \sin(\theta)$ ). The function  $h$  is smooth around the boundary of the disk, except at  $\theta = 0$  (same as  $\theta = 2\pi$ ) where  $h'(0) = 2\pi$  while  $h'(2\pi) = -2\pi$ . The harmonic function  $u(r, \theta)$  with  $u(1, \theta) = h(\theta)$  is  $C^\infty(D)$ , and can be extended to  $C^0(\bar{D})$ , but  $u$  cannot be extended to  $C^1(\bar{D})$ . If the latter were possible then  $h$  would be once differentiable on  $\partial D$ , but it’s not. So if we required the solution  $u$  to satisfy  $\Delta u = 0$  in  $\bar{D}$ , we’d need  $u \in C^2(\bar{D})$ , and there’d be no solution.

Here’s one last example concerning solution regularity and how it’s sometimes denoted. We showed (or actually, you will show on the next test) that if  $u$  is harmonic on an open region  $D$  then  $u \in C^\infty(D)$ . If  $D$  is a disk in  $\mathbb{R}^2$  then from the Poisson integral formula we can see that if the Dirichlet data  $h$  is continuous,  $u$  must be continuous up to and onto  $\partial D$ . Such a statement would typically be written as  $u \in C^\infty(D) \cap C^0(\bar{D})$ , to indicate infinite differentiability on the open region and a continuous extension to the closure of  $D$ .

**Exercise:** Consider solving the heat equation  $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$  for  $-1 < x < 1$  with boundary conditions  $u(-1, t) = u(1, t) = 0$  and initial condition  $u(x, 0) = |x|$ . What goes wrong if we actually require  $\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0$  AT time  $t = 0$ ?

## 3.2 Classifying PDE’s

We’ve studied three main PDE’s in this course: Laplace’s equation, the heat equation, and the wave equation. These are the “big three” PDE’s from classical physics, and their solutions behave quite differently.

Consider a general second order constant-coefficient homogeneous PDE in two variables (the following ideas extend to  $n$  variables and variable coefficients) which I’ll call  $x$  and  $y$

$$c_{11}u_{xx} + c_{12}u_{xy} + c_{22}u_{yy} + c_1u_x + c_2u_y + c_0u = 0.$$

Each such PDE fits into one of three categories, depending on the relative

values of  $c_{11}$ ,  $c_{12}$ , and  $c_{22}$ ; the coefficients for the lower order terms are irrelevant:

1. **Elliptic:** This is the case if  $c_{11}c_{22} - c_{12}^2 > 0$  (then the graph of the conic section  $c_{11}x^2 + c_{12}xy + c_{22}y^2 + c_1x + c_2y + c_0 = R$  is an ellipse). For example, Laplace's equation ( $c_{11} = c_{22} = 1, c_{12} = c_1 = c_2 = c_0 = 0$ ) is elliptic. Indeed, any equation of this form can be turned into Laplace's equation with a suitable change of variables.

Elliptic equations have solutions which are  $C^\infty$  inside the domain in which the PDE holds, and obey the maximum principle. The required boundary conditions are just like for Laplace's equation, i.e., you need to specify the function values or normal derivatives on the boundary of the domain.

2. **Parabolic:** This is the case if  $c_{11}c_{22} - c_{12}^2 = 0$  (then the graph of the conic section  $c_{11}x^2 + c_{12}xy + c_{22}y^2 + c_1x + c_2y + c_0 = R$  is a parabolic). In this case it's more intuitive to let  $t$  denote the second variable instead of  $y$ . The heat equation ( $c_{11} = -1, c_2 = 1, c_{22} = c_{12} = c_1 = c_0 = 0$ ) is parabolic.

Parabolic equations also have solutions which are smooth inside the domain of definition (typically something like  $x \in (a, b), t > 0$ ), and require boundary/initial conditions like the heat equation. The solutions obey the same maximum principle, and like the heat equation information propagates with infinite speed. Rough initial data is instantly smoothed out.

3. **Hyperbolic:** This is the case if  $c_{11}c_{22} - c_{12}^2 < 0$  (the conic section is a hyperbola), and in this case it's also more intuitive to let  $t$  denote the second variable instead of  $y$ . The wave equation ( $c_{11} = -1, c_{22} = 1, c_{12} = c_1 = c_2 = c_0 = 0$ ) is hyperbolic.

Again, hyperbolic equations have solution that are “wave-like”; in particular, some kind of causality holds so information travels at finite speed. Also, rough initial data gives rise to rough solutions for all time, in contrast to the parabolic case.

These classifications extend to PDE's with  $n$  variables, and even variable coefficients. But not every PDE fits this classification. It's more problematic to classify PDE's of other orders, or with nonlinearities.