

Models Using the Continuity Equation

Mathematical Modeling Week 9

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Continuity Equation Recap

If $\rho(x, t)$ is the density of some “stuff” in two dimensions (with $x = (x_1, x_2)$) and $\mathbf{q}(x, t) = \langle q_1(x, t), q_2(x, t) \rangle$ is the flux of that stuff over some two dimensional region then we have seen that conservation of stuff implies the continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} = 0 \quad (1)$$

where $\nabla \cdot \mathbf{q} = \frac{\partial q_1}{\partial x_1} + \frac{\partial q_2}{\partial x_2}$. More generally, if stuff isn’t conserved, but is produced or destroyed at a rate $k(x, t)$ (dimensions stuff per unit time per unit area) then the equation is

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{q} = k. \quad (2)$$

In the special case of steady-state flow we have

$$\nabla \cdot \mathbf{q} = k. \quad (3)$$

In this case $\nabla \cdot \mathbf{q}$ equals the rate of creation (or destruction) of the material.

Constitutive Relations

As in one dimension, we can’t solve for ρ or \mathbf{q} without at least one more equation relating these two unknowns. The following problems are designed to make you think about an appropriate second equation—a constitutive relation—in various physical situations.

Here are a couple of pieces of terminology you should know: A physical object or material is said to be *homogeneous* if its physical properties don’t depend on position; one part of the material is the same as any other part, so it’s uniform. If a material is not homogeneous then it is said to be *nonhomogeneous*. An object or material is said to be *isotropic* if its physical properties at any point don’t depend on the orientation of the object; the properties are “direction independent.” Otherwise, the material is *anisotropic*. For

example, to good approximation most metals are isotropic electrical conductors; the electrical properties at any point in the metal—say, the resistance to electrical current—are independent of the direction in which the current flows. On the other hand many types of crystals conduct better in some directions than others, so they’re electrically anisotropic. Muscle tissue is also an anisotropic conductor—it carries current much more easily in the direction of the muscle fiber than across the fibers. Whether a material is modeled as homogeneous or isotropic can also depend on the scale at which the modeling is done. Some things which look homogeneous or isotropic on a large scale are quite inhomogeneous or anisotropic at a smaller scale.

Exercises:

1. (Incompressible Flow) Suppose that some fluid—say water—flows over the two dimensional plane. We’ll measure water by mass, so ρ has units of mass per area and \mathbf{q} has units of mass per length per time. No water is created or destroyed, so $k = 0$ in the continuity equation. Water is essentially incompressible too, so $\rho_t \equiv 0$. What can you conclude about \mathbf{q} ? This result of course applies to the flow of any incompressible fluid.

Based on your answer, could $\mathbf{q}(x_1, x_2, t) = \langle x_1, x_2 \rangle$ represent the flow of water over the plane? Plot this vector field to confirm your answer. Note this vector field doesn’t depend on time (but there’s no reason it couldn’t in general).

Repeat for the vector fields $\mathbf{q}(x_1, x_2, t) = \langle tx_2, -tx_1 \rangle$ and $\mathbf{q}(x_1, x_2, t) = \langle c_1, c_2 \rangle$ (c_1 and c_2 some constants).

2. (Advection) Suppose that $\rho(x, t)$ represents the concentration (say mass per area) of pollutant dissolved in water flowing over the two-dimensional plane. The water flows around, carrying the pollutant with it. We’ll suppose the pollutant doesn’t diffuse.

Let $\mathbf{v}(x_1, x_2, t)$ denote the velocity field of the water as it flows around. From question (1) we know that $\nabla \cdot \mathbf{v} = 0$ at all points and times.

- (a) If $\mathbf{q}(x_1, x_2, t)$ is the flow-rate for the pollutant, how are \mathbf{q} , \mathbf{v} , and ρ related? Hint: You can figure it out on dimensional grounds, or actually think it through. In the latter case, consider a very short

line segment L oriented perpendicular to \mathbf{v} , and assume \mathbf{v} and ρ are essentially constant near the segment. How fast is pollutant crossing L on a “stuff per time per length” basis?

- (b) Use the continuity equation and the constitutive relation giving \mathbf{q} in terms of \mathbf{v} and ρ to write out the equation satisfied by ρ (it involves \mathbf{v} too). The equation simplifies quite a bit if you use the fact that $\nabla \cdot \mathbf{v} = 0$.
- (c) Consider the simple case in which $\mathbf{v} = \langle c_1, c_2 \rangle$ for constants c_1 and c_2 . Write out the equation satisfied by $\rho(x_1, x_2, t)$. If the initial condition is $\rho(x_1, x_2, 0) = \phi(x_1, x_2)$, write out the solution $\rho(x_1, x_2, t)$ explicitly in terms of ϕ, c_1 , and c_2 .
- (d) Suppose that $\mathbf{v} = \langle v_1, v_2 \rangle$ depends on x_1, x_2 , and t . A characteristic curve $(x_1(t), x_2(t))$ for the vector field \mathbf{v} is a solution to the coupled DE's

$$\begin{aligned} x'_1(t) &= v_1(x_1(t), x_2(t), t) \\ x'_2(t) &= v_2(x_1(t), x_2(t), t) \end{aligned}$$

Show that ρ is constant on such a characteristic. Hint: look at $\frac{d}{dt}(\rho(x_1(t), x_2(t), t))$ on the characteristic.

Use this to find $\rho(x_1, x_2, t)$ explicitly in the case in which $\mathbf{v}(x_1, x_2, t) = \langle x_2, -x_1 \rangle$ with initial condition $\rho(x_1, x_2, 0) = \phi(x_1, x_2)$.

3. (Heat and Diffusion)

- (a) Suppose that $\rho(x, t)$ represents the thermal energy density (proportional to temperature, so you can assume it's just temperature) in some object, and that this energy is conserved as it diffuses through the object. What would \mathbf{q} represent? What are its dimensions? Most importantly, if the object is homogeneous and isotropic, what is a reasonable constitutive relation between ρ and \mathbf{q} ? Hint: heat flows in the steepest temperature decrease, in proportion to the rate of temperature decrease. The constant of proportionality is the *diffusivity*. What units should diffusivity have here?

(b) Use the continuity equation and constitutive relation from above to show that ρ should satisfy

$$\frac{\partial \rho}{\partial t} - \kappa \left(\frac{\partial^2 \rho}{\partial x_1^2} + \frac{\partial^2 \rho}{\partial x_2^2} \right) = 0.$$

(c) Suppose the heat flows in some region D . The region starts off with some initial temperature $\phi(x_1, x_2)$. What's the corresponding initial condition for ρ ? Suppose that at the boundary D no heat can exit or enter. Explain why this forces $\mathbf{q} \cdot \mathbf{n} = 0$, where \mathbf{n} is the outward unit normal vector on the boundary. Translate $\mathbf{q} \cdot \mathbf{n} = 0$ into an equation involving u .