

The Calculus of Variations

MA 348

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Image Reconstruction Review

Recall the image restoration problem we considered a couple weeks ago. An “image” in 1D can be considered as a function $a(t)$ defined on some interval, say $0 \leq t \leq 1$. We sample $a(t)$ at points $t_i = i/n$ for $i = 1$ to n , to obtain a discretized image $a_i = a(t_i)$. But a_i is corrupted by noise, so we really measure $y_i = a_i + \epsilon_i$ where ϵ_i is some kind of random noise. We attempted to recover the actual image by minimizing

$$f(\mathbf{x}) = \frac{1}{2} \sum_{i=1}^n (x_i - y_i)^2 + \frac{\alpha}{2} \sum_{i=1}^{n-1} \left(\frac{x_{i+1} - x_i}{h} \right)^2 \quad (1)$$

where $h = 1/n$ and α is a parameter at our disposal. The minimizer \mathbf{x} is the reconstructed image. The idea was that the first sum on the right in equation (1) encourages $x_i \approx y_i$, while the second sum is a term to penalize rapid variation in the reconstruction. The scheme had the disadvantage of blurring crisp edges in an image, so we considered by alternate approach of using a penalty term of the form $\sum_i \frac{|x_{i+1} - x_i|}{h}$.

Consider multiplying the right side of equation (1) by h and then letting n approach infinity. If you think of $x_i = x(t_i)$ and $y_i = y(t_i)$ for some functions $x(t)$ and $y(t)$, then the right side approaches limit $\frac{1}{2} \int_0^1 (x(t) - y(t))^2 dt + \frac{\alpha}{2} \int_0^1 (x'(t))^2 dt$, at least if $x(t)$ and $y(t)$ are nice enough functions. This is the continuous version of the image restoration problem: given a function $y(t)$, find a function $x(t)$ that minimizes the quantity

$$J(x) = \frac{1}{2} \int_0^1 (x(t) - y(t))^2 dt + \frac{\alpha}{2} \int_0^1 (x'(t))^2 dt. \quad (2)$$

The Calculus of Variations

The calculus of variations is about finding functions that minimize integrals. This might sound like a useless problem, but it’s actually one of the most important areas of classical and modern applied math. As you can see above, the image restoration problem can be cast as a calculus of variations problem. The solutions to most PDE’s can also be found by a minimization process, and this leads to finite element methods. This is also the basis of much of optimal control theory, and the calculus of variations is part of the mathematics behind the Lagrangian and Hamiltonian formulations of the laws of physics.

To illustrate the general idea, let’s consider a simple model problem. We want to find a function which minimizes the integral

$$Q(\phi) = \int_0^1 (\phi'(x))^2 dx \quad (3)$$

with the additional restriction that $\phi(0) = 2$ and $\phi(1) = 4$. How do we go about doing this?

Pretend that you're a junior in high school again; you've studied precalculus mathematics, so you know what functions are, but you've never seen a derivative. Someone asks you to find a number x_0 which minimizes the function $f(x) = x^4 - 3x + 7$. You know what that means: the minimizer x_0 will be a number for which the functional value is lower than that for any nearby number. In other words

$$f(x_0) \leq f(x_0 + \epsilon)$$

for any ϵ (if we require $f(x_0) \leq f(x_0 + \epsilon)$ only for sufficiently small ϵ we get a LOCAL minimum). If you expand the above inequality out and do some simple algebra you obtain

$$(4x_0^3 - 3)\epsilon + 6x_0^2\epsilon^2 + 4x_0\epsilon^3 + \epsilon^4 \geq 0 \quad (4)$$

where the powers of ϵ have been grouped together. Think of x_0 as fixed and consider the left side of inequality (4) as a function of ϵ . How can the left side be positive for ALL ϵ ? If $|\epsilon|$ is sufficiently close to zero then the ϵ^2 and higher powers are negligible compared to ϵ , and so the sign of the left side of inequality (4) will be determined by the coefficient of ϵ . Now if $4x_0^3 - 3 > 0$ then choosing $\epsilon < 0$ and small will violate the inequality (4). If $4x_0^3 - 3 < 0$ then choosing $\epsilon > 0$ and small will again violate the inequality (4). The only way that (4) can hold for all small ϵ is if $4x_0^3 - 3 = 0$, so that the ϵ^2 term dominates. But if $4x_0^3 - 3 = 0$ then we can solve to find $x_0 = \sqrt[3]{3/4}$ (this is the only real root). This is a necessary condition for x_0 to be a local minimum (but not sufficient). To see if this is sufficient, you'd need to look at the ϵ^2 term.

The same procedure allows us to find the minimizer of $Q(\phi)$ defined in equation (3). Let's use f to denote that function which minimizes Q . Just as we perturbed x_0 above by adding in a small number ϵ , we will perturb the minimizer f by adding in a small function. Let $\eta(x)$ be any nice differentiable function with $\eta(0) = \eta(1) = 0$. Then if f really is the minimizer of Q (with the right boundary conditions) then

$$Q(f + \epsilon\eta) \geq Q(f) \quad (5)$$

for ANY small number ϵ , and ANY function η with $\eta(0) = \eta(1) = 0$. Why must we require that $\eta(0) = \eta(1) = 0$? Because otherwise the function $f(x) + \epsilon\eta(x)$ doesn't satisfy the boundary conditions and so wouldn't be a legitimate contender for the minimizer. Expand out both sides of (5) and do a bit of algebra to find that

$$2\epsilon \int_0^1 f'(x)\eta'(x) dx + \epsilon^2 \int_0^1 (\eta'(x))^2 dx \geq 0. \quad (6)$$

As before, the only way equation (6) can hold for all ϵ is if the coefficient of ϵ is zero. We conclude that

$$\int_0^1 f'(x)\eta'(x) dx = 0. \quad (7)$$

Now integrate equation (7) by parts with $u = f$ and $dv = \eta' dx$. With the boundary conditions on η we obtain

$$\int_0^1 f''(x)\eta(x) dx = 0. \quad (8)$$

Now remember, η was ANY differentiable function with $\eta(0) = \eta(1) = 0$. How can equation (8) be true for any such η ? You should be able to convince yourself that equation (8) can hold for all η ONLY IF $f''(x) = 0$ on the interval $(0, 1)$. But this means that $f(x)$ is of the form $f(x) = c_1x + c_2$, and if it's to satisfy $f(0) = 2$ and $f(1) = 4$ then f must be $f(x) = 2x + 2$. This function is the minimizer of Q .

Let's look at a few more examples:

Example 2: Consider minimizing

$$Q(\phi) = \int_0^1 ((\phi'(x))^2 + \phi^2(x)) dx$$

with the conditions that $\phi(0) = 1$, but with no other restrictions on ϕ . The minimizer f will satisfy

$$Q(f) \leq Q(f + \epsilon\eta)$$

for all C^1 functions η with $\eta(0) = 0$ and real numbers ϵ . Note here that because we have no restrictions at the endpoint $x = 1$, there are no restrictions on $\eta(1)$. If you expand out the above equation you find that

$$\epsilon(2 \int_0^1 (f(x)\eta(x) + f'(x)\eta'(x)) dx) + O(\epsilon^2) \geq 0.$$

Reasoning as before, this can hold only if

$$\int_0^1 (f(x)\eta(x) + f'(x)\eta'(x)) dx = 0.$$

Integrate the second term by parts and use the fact that $\eta(0) = 0$ to find that

$$\int_0^1 (f(x) - f''(x))\eta(x) dx + f'(1)\eta(1) = 0$$

for all functions η with $\eta(0) = 0$. Now reason as before: if $f(x) - f''(x) \neq 0$ for some x in $(0, 1)$ then we can certainly find a function η (and can arrange $\eta(1) = 0$, too) so that the integral above is not zero. We conclude that $f - f''$ is identically zero on the interval $(0, 1)$. If this is true then the above equation becomes $f'(1)\eta(1) = 0$, and since we can arrange for $\eta(1)$ to be anything we like, we must conclude that $f'(1) = 0$. So in the end the minimizer f must satisfy

$$f(x) - f''(x) = 0$$

with $f(0) = 1$ and $f'(1) = 0$. The general solution to $f - f'' = 0$ is $f(x) = c_1 e^x + c_2 e^{-x}$. To arrange the boundary conditions we solve two equations in two unknowns, $c_1 + c_2 = 1$, $c_1 e - c_2/e = 0$ to find $c_1 = 1/(e^2 + 1)$, $c_2 = e^2/(e^2 + 1)$. The minimizer is

$$f(x) = \frac{1}{e^2 + 1} (e^x + e^{2-x}).$$

Example 3: Let's prove that the shortest path between two points is a straight line. Let the points be (x_0, y_0) and (x_1, y_1) . Let's also make the assumption that the path can be described as the graph of a function $y = f(x)$. Then what we seek to minimize is

$$Q(\phi) = \int_{x_0}^{x_1} \sqrt{1 + (\phi'(x))^2} dx$$

subject to the restrictions that $\phi(x_0) = y_0$ and $\phi(x_1) = y_1$. As before, we know that the shortest path f will satisfy $Q(f) \leq Q(f + \epsilon\eta)$ where $\eta(x_0) = 0$ and $\eta(x_1) = 0$, and ϵ is any number. Written out in detail this is just

$$0 \leq \int_{x_0}^{x_1} (\sqrt{1 + (f'(x))^2 + 2\epsilon f'(x)\eta'(x) + \epsilon^2(\eta'(x))^2} - \sqrt{1 + (f'(x))^2}) dx. \quad (9)$$

Looks like a mess. We can simplify by using the fact that if ϵ is a small number then

$$\sqrt{a + \epsilon b} = \sqrt{a} + \frac{b}{2\sqrt{a}}\epsilon + O(\epsilon^2).$$

This comes from the tangent line or Taylor series approximation. Apply this to the integrand in inequality (9) with $a = 1 + (f'(x))^2$ and $b = 2f'\eta' + \epsilon(\eta')^2$. You find that the first messy square root is

$$\sqrt{1 + (f'(x))^2 + 2\epsilon f'(x)\eta'(x) + \epsilon^2(\eta'(x))^2} = \sqrt{1 + (f'(x))^2} + \epsilon \frac{f'\eta'}{\sqrt{1 + (f')^2}} + O(\epsilon^2).$$

Put this into the integral in (9), do the obvious cancellations, and note that as before we need the ϵ term to be zero if the inequality is to hold up. This gives

$$\int_{x_0}^{x_1} \frac{f'(x)\eta'(x)}{\sqrt{1 + (f'(x))^2}} dx = 0$$

for all functions η with $\eta(x_0) = \eta(x_1) = 0$. Integrate by parts in x to get the derivative terms off of η . It's messy, but in the end you get (using the endpoint conditions on η)

$$\int_{x_0}^{x_1} \eta(x) \frac{f''(x)}{(1 + (f'(x))^2)^{3/2}} dx = 0.$$

As before, for this to be true we need $\frac{f''(x)}{(1 + (f'(x))^2)^{3/2}} = 0$. Now, the denominator is always positive, so for this to be true we need $f''(x) = 0$, i.e., $f(x) = mx + b$, a line. Of course, m

and b are adjusted to make it go through the two points (x_0, y_0) and (x_1, y_1) .

General Principles: The previous examples lead us to some general principles for solving calculus of variations problems. I don't want to give a very specific recipe, just some general guidelines. To minimize $Q(\phi)$.

1. Start with the inequality

$$Q(f) \leq Q(f + \epsilon\eta)$$

where f is the minimizer to be found and η is a legitimate test function to put into Q .

2. One way or another, manipulate the inequality above into something of the form

$$0 \leq \epsilon V_1 + O(\epsilon^2)$$

where V_1 is typically an integral involving f and η . The quantity V_1 is called the *first variation* of Q , and plays the role similar to that of a first derivative. For the above inequality to hold for all ϵ we need $V_1 = 0$.

3. Manipulate V_1 , often using integration by parts, to determine f . This usually leads to an equation like

$$\int L(f)\eta \, dx = 0$$

where $L(f)$ is some kind of differential equation (ordinary or partial) satisfied by f . The only way this integral can be zero is for $L(f) = 0$. Find f by solving the DE.

One thing to note: in each case above we found a critical point, that is, a function which satisfies a necessary condition for a minimum, but that's not the same as sufficient. Just as in Calculus I, when you find a critical point you have to do some additional analysis to see if the tentative critical point is really a minimum. There are such tests in the calculus of variations (something like a second derivative test), but I won't go into them here.

Problem

- The image restoration problem is often cast as finding a function $x(t)$ which minimizes

$$J(x) = \frac{1}{2} \int_0^1 (x(t) - y(t))^2 \, dt + \frac{\alpha}{2} \int_0^1 \phi(x'(t)) \, dt$$

where the user chooses ϕ . One might take $\phi(u) = u^2$, leading to the edge blurring scheme. Or perhaps $\phi(u) = |u|$, which preserves edges but is not differentiable (so we have a slight problem). We can also try $\phi(u) = \sqrt{u^2 + \delta}$ for some small δ , to get a differentiable ϕ that approximates the absolute value.

Work out the second order DE that the minimizer for $J(x)$ above must satisfy; the DE involves ϕ and its derivatives. Write out the specific DE obtained when $\phi(u) = u^2$ and when $\phi(u) = \sqrt{u^2 + \delta}$. You can assume boundary conditions $x'(0) = x'(1) = 0$.

Then construct a piecewise constant function $y(t)$ and solve the DE for each ϕ (numerically, in Maple probably) and compare the reconstruction to the original y .