The $25,000,000,000 Eigenvector
The Linear Algebra Behind Google

Kurt Bryan

May 19, 2011

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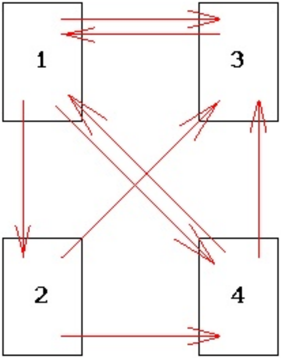
An essential factor in this ordering is the importance of each web page. How should "importance" be computed?

A Simple Idea



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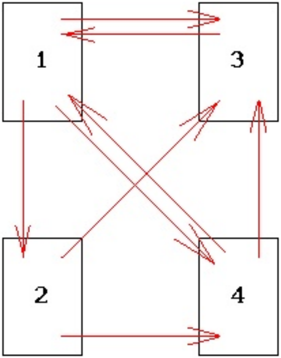

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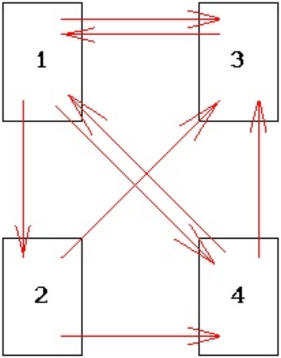

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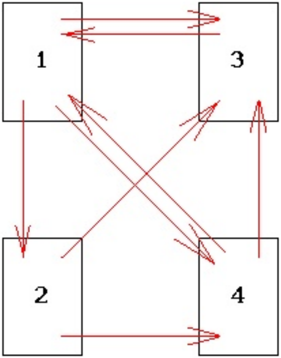

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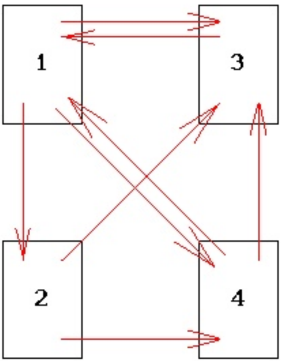

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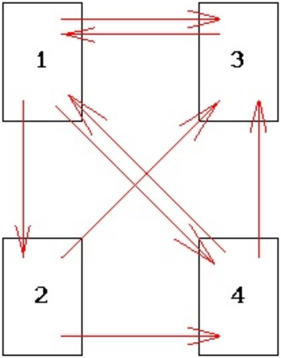
- $x_3 = 3$ (most important)
- $x_4 = 2$

A Shortcoming



Backlink counting yields $x_1 = 2, x_2 = 1, x_3 = 3, x_4 = 2.$

But shouldn't $x_1 > x_4$?

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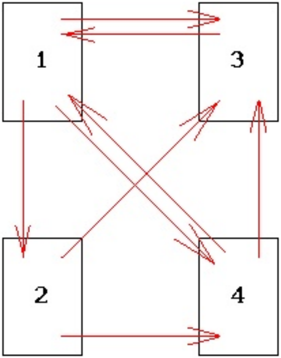

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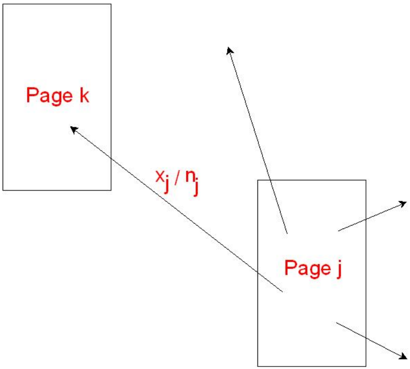
Better Idea: Links from important pages should count more than links from less important pages.

An Improvement



If page j links to page k then page j casts a vote for page k's importance, in amount x_j/n_j, where n_j is the number of links out of page j.

The importance score for any page is the sum of its votes from its backlinks.

$$x_k = \cdots + x_j/n_j + \cdots$$

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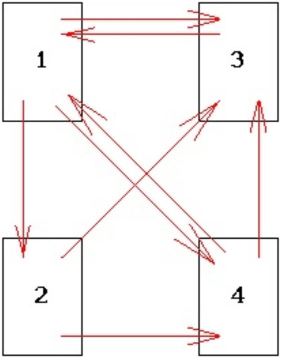

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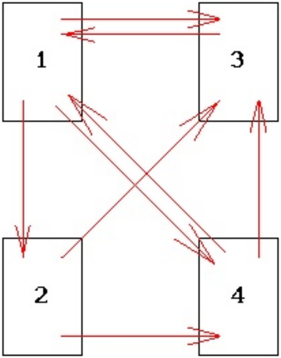

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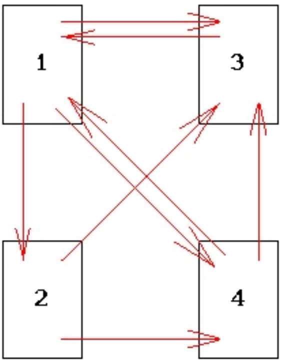

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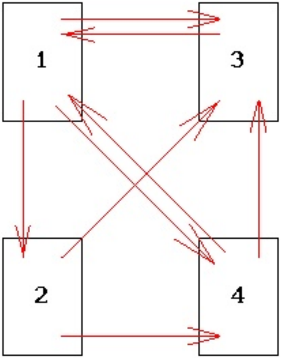

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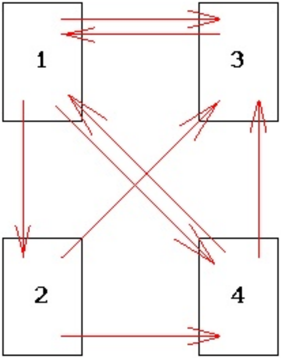

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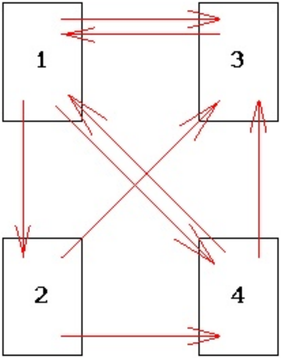
The Matrix Form



If $\mathbf{x} = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]^T$ then

$$\mathbf{x} = \mathbf{A}\mathbf{x}$$

where $\mathbf{A}$ is the link matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 1/2 \\ 1/3 & 0 & 0 & 0 \\ 1/3 & 1/2 & 0 & 1/2 \\ 1/3 & 1/2 & 0 & 0 \end{bmatrix}$$

So $\mathbf{x}$ is an eigenvector for $\mathbf{A}$ with eigenvalue 1.

Eigenvectors

Recall that if $\mathbf{A}$ is an $n \times n$ matrix then a NONZERO vector $\mathbf{x}$ is an eigenvector for $\mathbf{A}$ if

$$\mathbf{A}\mathbf{x} = \lambda \mathbf{x}$$

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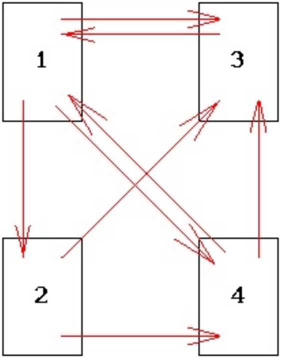
- n corresponding linearly independent eigenvectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$
- Any nonzero multiple of an eigenvector is again an eigenvector.

The Importance Eigenvector



Such an eigenvector exists here:

$\mathbf{x} \approx [0.387, 0.129, 0.290, 0.194]^T$

(or any multiple) satisfies
$\mathbf{x} = \mathbf{A}\mathbf{x}$.

Such an eigenvector must ALWAYS exist if $\mathbf{A}$ is *column-stochastic* (columns of $\mathbf{A}$ sum to one) and there are no dangling nodes.

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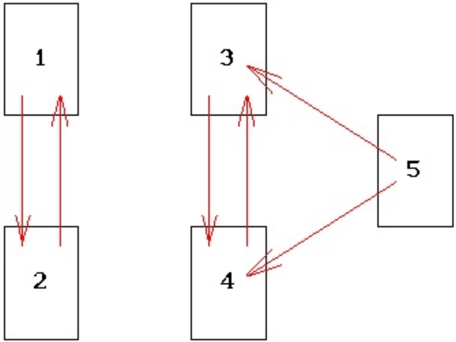
- Should be "easy" to compute for an eight billion by eight billion matrix.

Problems



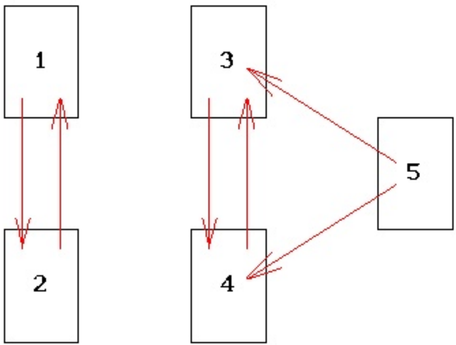
The link matrix here is

$$\mathbf{A} = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1/2 \\ 0 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Two-dimensional eigenspace, basis $\mathbf{x}_1 = [1/2, 1/2, 0, 0, 0]^T$, $\mathbf{x}_2 = [0, 0, 1/2, 1/2, 0]^T$.

Any linear combination $c_1\mathbf{x}_1 + c_2\mathbf{x}_2$ is an eigenvector too!

Web With Multiple Components



Which eigenvector should we use?

Notation: $V_1(\mathbf{A})$ is the subspace of eigenvectors for $\mathbf{A}$ with eigenvalue 1.

Theorem: If a web has r components (considered as a graph) then $V_1(\mathbf{A})$ has at least dimension r.

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- Replace n by n matrix $\mathbf{A}$ with weighted combination

$$\mathbf{M} = (1 - m)\mathbf{A} + m\mathbf{S}$$

where $0 \leq m \leq 1$ and $\mathbf{S}$ is n by n,

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- Then $\mathbf{M}$ is column-stochastic. Original problem is $m = 0$, while $m = 1$ is egalitarian web—all pages equal importance.

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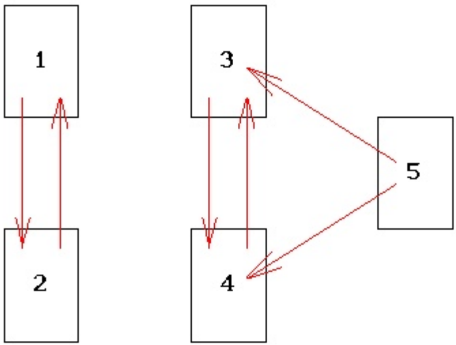
- Google (reputedly) uses $m = 0.15$.
- All we need is a method for finding an eigenvector for an eight billion by eight billion matrix!

Multiple Component Example



With $m = 0.15$ the link matrix is

$$\begin{bmatrix} 0.03 & 0.88 & 0.03 & 0.03 & 0.03 \\ 0.88 & 0.03 & 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 & 0.88 & 0.46 \\ 0.03 & 0.03 & 0.88 & 0.03 & 0.46 \\ 0.03 & 0.03 & 0.03 & 0.03 & 0.03 \end{bmatrix}$$

The importance eigenvector is
$\mathbf{x} =$
$[0.2, 0, 2, 0.285, 0.285, 0.03]^T$.

Interesting observation: As
$m \to 0^+$,
$\mathbf{x} \to [0.2, 0, 2, 0.3, 0.3, 0.0]^T$.

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In short, $\mathbf{M}^k\mathbf{x}_0$ will converge to the eigenvector we want, for “any” initial guess $\mathbf{x}_0$.

Power Method Example

With the **M** for the five page network above we find

$$\mathbf{x}_0 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.2 \\ 0.2 \\ 0.4 \end{bmatrix}, \quad \mathbf{M}^{20}\mathbf{x}_0 = \begin{bmatrix} 0.196 \\ 0.196 \\ 0.289 \\ 0.289 \\ 0.03 \end{bmatrix}, \quad \mathbf{M}^{40}\mathbf{x}_0 = \begin{bmatrix} 0.199 \\ 0.199 \\ 0.286 \\ 0.286 \\ 0.03 \end{bmatrix},$$

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- For **M** we have $\lambda_1 = 1$ and it can be shown that $\lambda_2 = 1 - m$. If $m = 0.15$ the method converges in proportion to $(0.85)^k$.
- But **M** is dense (mostly non-zero). Computing the matrix-vector product **Mv** requires about 1.3×10^{20} operations for a web with eight billion pages!

Convergence and Implementation

- $\mathbf{Mv}$ can be computed efficiently: $\mathbf{M} = (1 - m)\mathbf{A} + m\mathbf{S}$, where $\mathbf{S}$ has all entries $1/n$ and $\mathbf{A}$ is sparse (mostly zeros), since most web pages link to only a few other pages.

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- If each page has, on average, 10 outgoing links, $\mathbf{A}$ has only 10 non-zero entries per column. $\mathbf{Av}$ can be computed in about 1.6×10^{11} operations.
- Then $\mathbf{Mv} = (1-m)\mathbf{Av} + m\mathbf{Sv} = (1-m)\mathbf{Av} + [m/n, m/n, \ldots, m/n]^T$ can be computed in a reasonable time.

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- Sophisticated linear algebra is at the core of much scientific computation, and pops up when you least expect it.
- If you pay attention to your math professors, you might become a billionaire.

Resources

1. *The $25,000,000,000 Eigenvector: The Linear Algebra Behind Google*, SIAM Review (Education Section), Vol 48 (3), August 2006.
2. `www.rose-hulman.edu/~bryan/google.html`