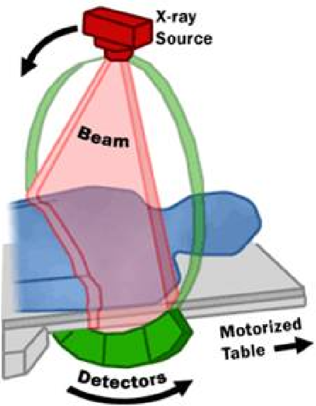
Inverse Problems 4: The Mathematics of CT Scanners

Kurt Bryan

April 28, 2011

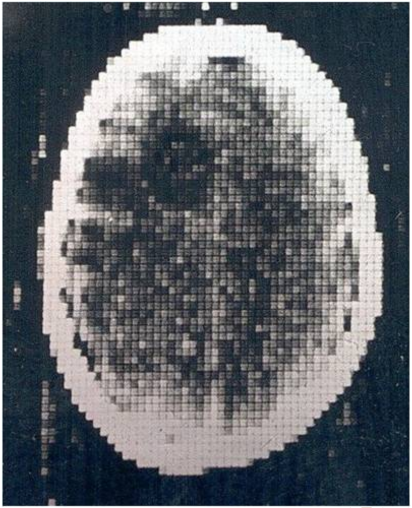
First CT Scanners

First practical scanners built in the late 1960's.



First CT Scanners

Images took hours to process/render, and were crude:



Modern CT Scanners

Modern scanners are fast and high-resolution:

The Mathematics

The mathematics underlying the model for a CT scanner is much older.

- Based on the Radon (and Fourier) transforms, dating back the early 20th century (and farther).

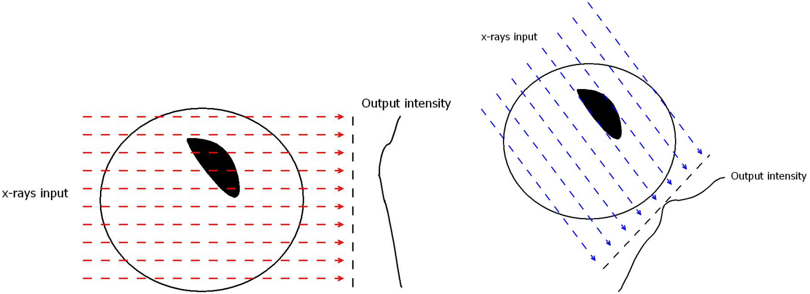
The Mathematics

The mathematics underlying the model for a CT scanner is much older.

- Based on the Radon (and Fourier) transforms, dating back the early 20th century (and farther).
- Most of it is easy enough to do in a Calc 2 class!

Mathematical Model

We fire x-rays through a body at many angles and offsets, measure beam attenuation (output/input intensity):



Attenuation of x-rays

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- Let $I(s)$ be the intensity of the x-ray along $\mathbf{L}$, with $I(a) = I_a$ (known input intensity).
- We suppose the x-ray beam is attenuated according to

$$I'(s) = -\lambda(\mathbf{L}(s))I(s)$$

as it passes through the body. The function λ is called the *attenuation coefficient*. We want to find λ.

Attenuation of x-rays

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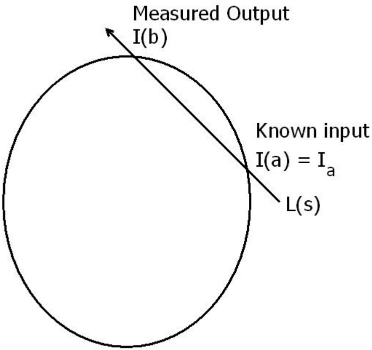

We measure the output $I(b)$.

Solving the Attenuation DE

The DE $I'(s) = -\lambda(\mathbf{L}(s))I(s)$ with $I(a) = I_a$ is easy to solve via separation of variables. We find

$$I(s) = I_a \exp\left(-\int_a^s \lambda(\mathbf{L}(t))\,dt\right).$$

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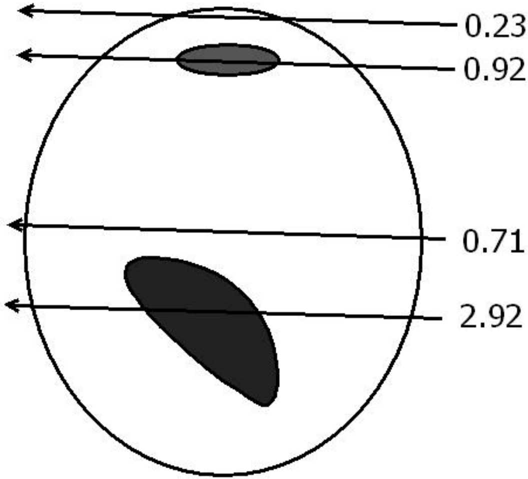
If we know (measure) $I(b)$ then we can compute

$$\int_a^b \lambda(\mathbf{L}(t))\,dt = -\ln(I(b)/I(a)).$$

We can find the integral on the left, for any line through the body.

Attenuation Example

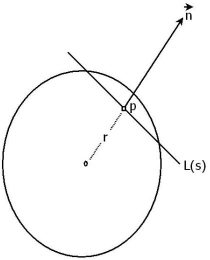
Some line integrals:



Geometry and Notation

Suppose $\mathbf{L}(s) = \mathbf{p} + s\mathbf{n}^{\perp}$, where

- $\mathbf{n} =< \cos(\theta), \sin(\theta) >$ dictates line normal vector, $\theta \in [0, \pi)$.
- $\mathbf{p} = r\mathbf{n}$, $r \in (-1, 1)$ is "offset" from the origin.



Note $-\sqrt{1 - r^2} < s < \sqrt{1 - r^2}$.

The Radon Transform

In summary, by firing x-rays through the body, we can compute the integral

$$d(r,\theta) = \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \lambda(\mathbf{L}(s))\, ds$$

for $0 \leq \theta < \pi, -1 < r < 1$.

The quantity $d(r,\theta)$ is called the "Radon Transform" of λ.

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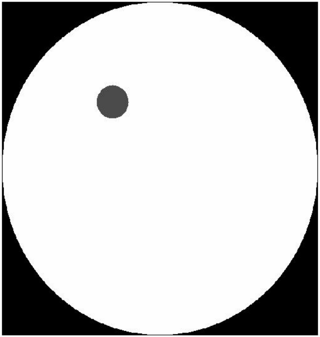
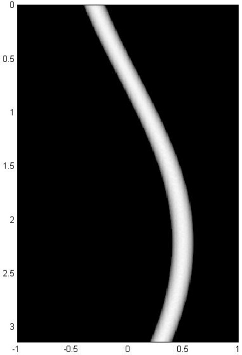
Is this enough to determine λ? How?

The Sinogram

CT target and its sinogram:





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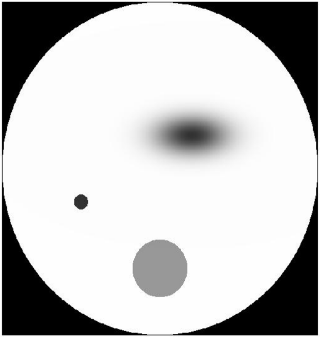

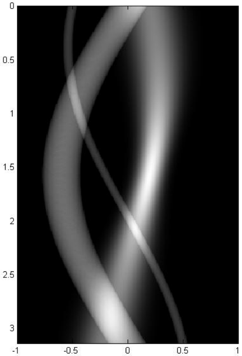

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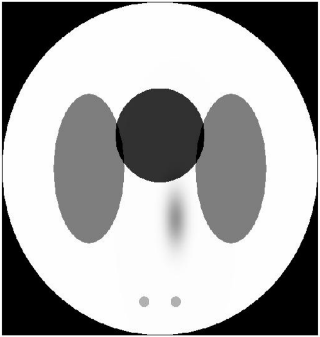
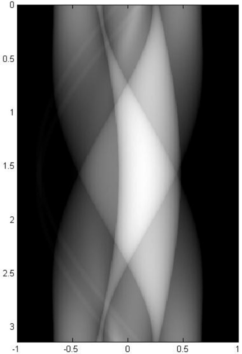
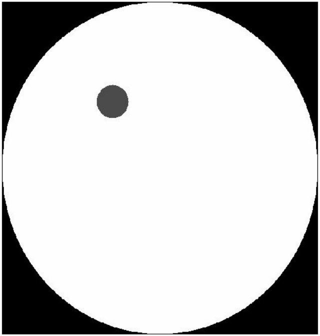
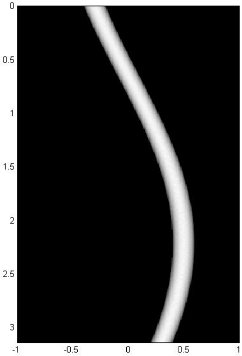




Intuition

Observation: Every x-ray through a high attenuation region will yield a large line integral.



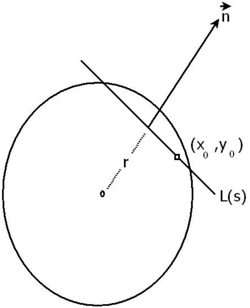


Intuition Quantified

For any fixed point (x_0, y_0) in the body, the line $\mathbf{L}(s)$ with normal at angle θ is given non-parametrically by

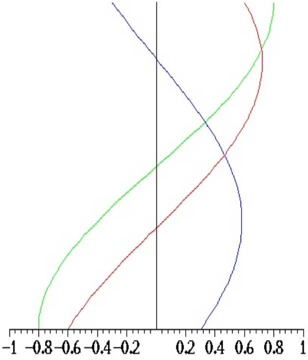
$$x\cos(\theta) + y\sin(\theta) = r$$

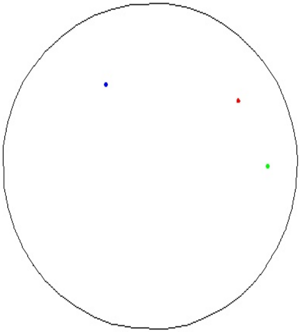
with $r = x_0\cos(\theta) + y_0\sin(\theta)$:



Intuition Quantified

Each point on the curve $r = x_0 \cos(\theta) + y_0 \sin(\theta)$ in the sinogram corresponds to a line through (x_0, y_0) in the target.





Intuition Quantified

The average value of the Radon transform $d(\theta, r)$ over all lines through (x_0, y_0) is then

$$\tilde{\lambda}(x_0, y_0) = \int_0^{\pi} d(\theta, x_0 \cos(\theta) + y_0 \sin(\theta))\, d\theta.$$

This is called the *backprojection* of $d(\theta, r)$.

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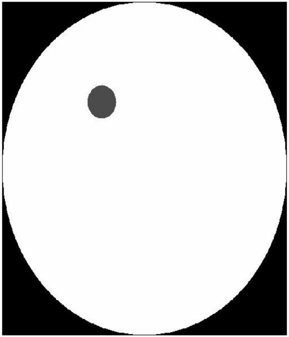
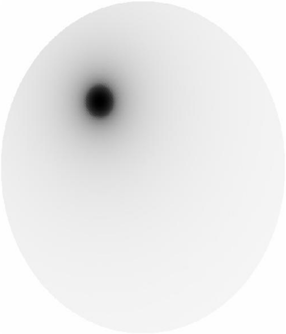
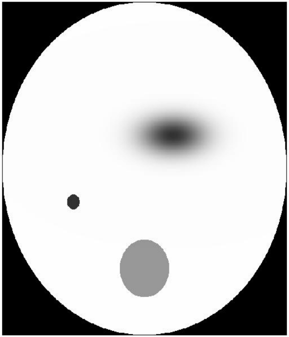
Maybe $\tilde{\lambda}$ will look like λ.

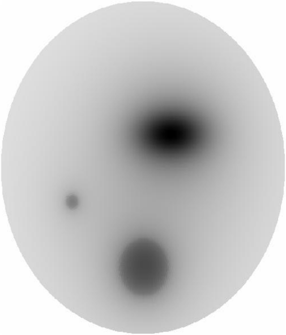
Unfiltered Backprojection Example 1



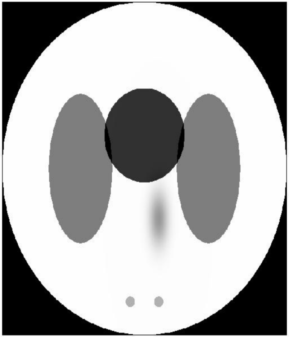


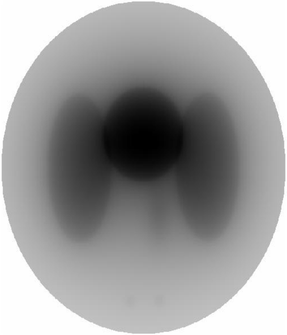
Unfiltered Backprojection Example 2





Unfiltered Backprojection Example 3





Unfiltered Backprojection is Blurry

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- Unfiltered backprojection is only an approximate inverse for the Radon transform.
- There's another step needed to compute the true inverse (and get sharper images).

Filtered Backprojection

If $d(\theta, r)$ is the "raw" sinogram, first construct $\tilde{d}(\theta, r)$ by expanding into a Fourier series[1] with respect to r:

$$d(\theta, r) = \sum_{k=-\infty}^{\infty} c_k e^{i\pi kr} \text{ with } c_k = \int_{-1}^{1} d(\theta, r) e^{-i\pi kr}\, dr,$$

then

[1] in the continuous case, a Fourier integral transform

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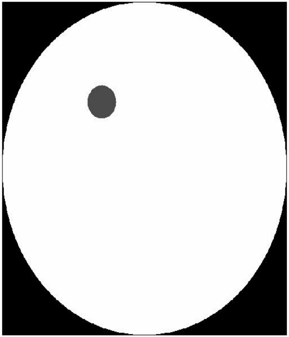
then set

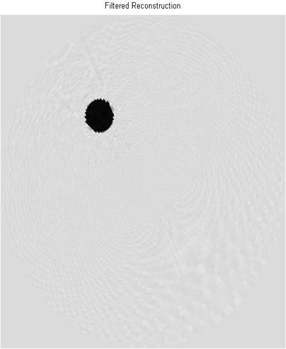
$$\tilde{d}(\theta, r) \sum_{k=-\infty}^{\infty} |k| c_k e^{i\pi kr}.$$

In signal processing terms, we apply a high-pass "ramp" filter to d, in the r variable. Finally, backproject.

[1]in the continuous case, a Fourier integral transform

Filtered Backprojection Example 1





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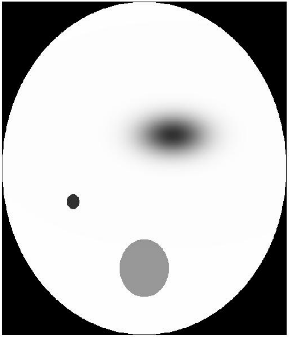

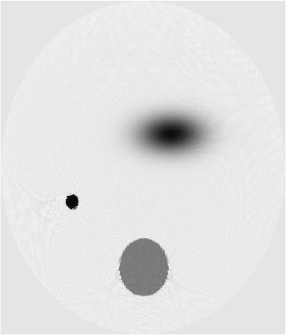

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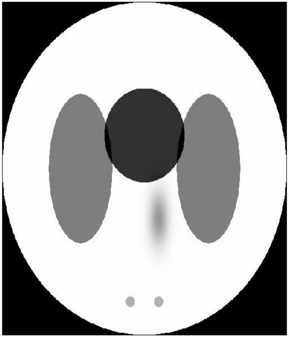
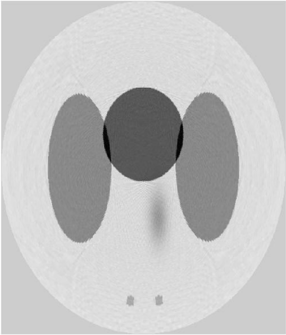
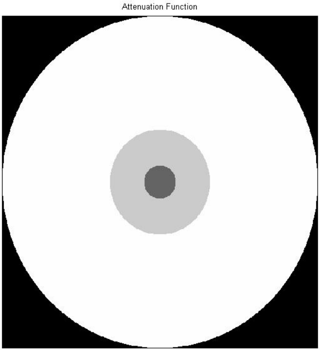




The Radial Case

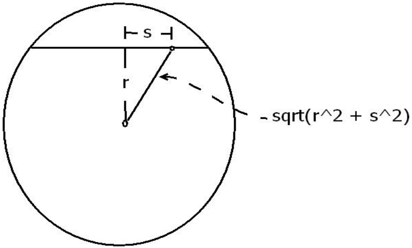
Suppose λ depends only distance from the origin, so $\lambda = \lambda(\sqrt{x^2 + y^2})$:



The Radial Case

In this case it's easy to see that the Radon transform depends only on r, not θ. For a line at distance r from the origin

$$d(r) = \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \lambda(\sqrt{r^2+s^2})\, ds = 2\int_0^{\sqrt{1-r^2}} \lambda(\sqrt{r^2+s^2})\, ds.$$



The Radial Case

In summary, if we are given the function $d(r)$ for $0 \leq r \leq 1$

$$d(r) = 2\int_0^{\sqrt{1-r^2}} \lambda(\sqrt{r^2+s^2})\, ds$$

can we find the function λ?

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With a couple change of variables, this integral equation can be massaged into a "well-known" integral equation.

The Radial Case

Start with

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$$d(r) = 2\int_0^{\sqrt{1-r^2}} \lambda(\sqrt{r^2+s^2})\,ds.$$

Make u-substitution $u = \sqrt{r^2+s^2}$, (so $s = \sqrt{u^2-r^2}$, and $ds = u\,du/\sqrt{u^2-r^2}$), obtain

$$d(r) = 2\int_r^1 \frac{u\lambda(u)}{\sqrt{u^2-r^2}}\,du.$$

The Radial Case

Rewrite as

$$
\begin{aligned}
d(r) &= 2\int_r^1 \frac{u\lambda(u)}{\sqrt{u^2 - r^2}}\, du \\
&= 2\int_r^1 \frac{u\lambda(u)}{\sqrt{(1-r^2)-(1-u^2)}}\, du.
\end{aligned}
$$

The Radial Case

Rewrite as

$$\begin{aligned} d(r) &= 2\int_r^1 \frac{u\lambda(u)}{\sqrt{u^2-r^2}}\,du \\ &= 2\int_r^1 \frac{u\lambda(u)}{\sqrt{(1-r^2)-(1-u^2)}}\,du. \end{aligned}$$

Define $z = 1 - r^2$ (so $r = \sqrt{1-z}$), substitute $t = 1 - u^2$ (so $u = \sqrt{1-t}, du = -\frac{1}{2\sqrt{1-t}}\,dt$) to find

$$d(\sqrt{1-z}) = \int_0^z \frac{\lambda(\sqrt{1-t})}{\sqrt{z-t}}\,dt.$$

The Radial Case

In

$$d(\sqrt{1-z}) = \int_0^z \frac{\lambda(\sqrt{1-t})}{\sqrt{z-t}}\,dt$$

define $g(z) = d(\sqrt{1-z})$ and $\phi(t) = \lambda(\sqrt{1-t})$. We obtain

$$\int_0^z \frac{\phi(t)}{\sqrt{z-t}}\,dt = g(z)$$

known as *Abel's* equation. It has a closed-form solution!

The Radial Case Solution

The solution to

$$\int_0^z \frac{\phi(t)}{\sqrt{z-t}}\,dt = g(z)$$

is

$$\phi(t) = \frac{1}{\pi}\frac{d}{dt}\left(\int_0^t \frac{g(w)\,dw}{\sqrt{t-w}}\right).$$

(Recall $g(w) = d(\sqrt{1-w})$). We solve for $\phi(t)$ and recover $\lambda(r) = \phi(1-r^2)$.

Radial Example

Suppose $d(r) = \frac{\sqrt{1-r^2}}{3}(14 + 4r^2)$. Then

$$g(z) = d(\sqrt{1-z}) = \frac{\sqrt{z}}{3}(18 - 4z)$$

and

$$\phi(t) = \frac{1}{\pi}\frac{d}{dt}\left(\int_0^t \frac{g(w)\,dw}{\sqrt{t-w}}\right) = 3 - t.$$

Finally

$$\lambda(r) = \phi(1 - r^2) = 2 + r^2.$$

Notes URL

www.rose-hulman.edu/~bryan/invprobs.html