Introduction to Inverse Problems II

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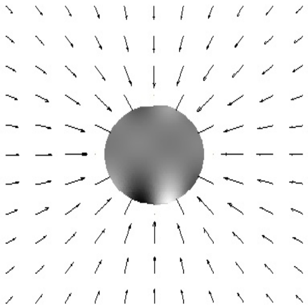
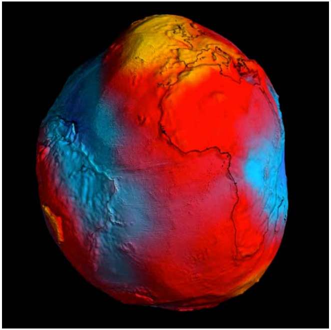
The Earth

Earth's gravitational field is not perfectly spherical, because the earth is not of uniform density (nor perfectly spherical).



The Earth

A rendition of the gravitational variation based on data from the Goce (Gravity Field and Steady-State Ocean Circulation Explorer) satellite; measurements accurate to one part per trillion!



http://www.bbc.co.uk/news/science-environment-12911806, March 31 2011.

The Goce Satellite

This picture has no intellectual content, but it looks pretty cool.

The Forward and Inverse Problems

Forward Problem: Given the density $\rho(x, y, z)$ of the earth, compute the gravitational field $\mathbf{F}(x_0, y_0, z_0)$ at some point in space.

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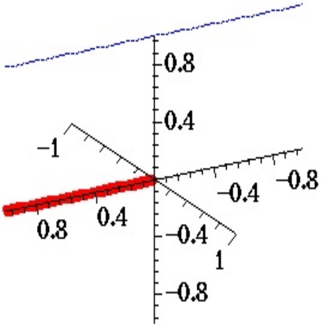
Inverse Problem: Given measurements of the gravitational field $\mathbf{F}(x_0, y_0, z_0)$, find the density function $\rho(x, y, z)$.

A 1D Version: The Forward Problem

A "1D" bar, length one meter, stretching along the x-axis from $0 < x < 1$ in 3D space, density $\lambda^*(x)$ Kg per meter of length:



A 1D Version: The Forward Problem

A short dx piece of the bar at position $(x, 0, 0)$ has mass $\lambda^*(x)\,dx$ and gravitational field

$$d\mathbf{F}(x_0, y_0, z_0) = \frac{G < x - x_0, -y_0, -z_0 > \lambda^*(x)\,dx}{((x - x_0)^2 + y_0^2 + z_0^2)^{3/2}}.$$

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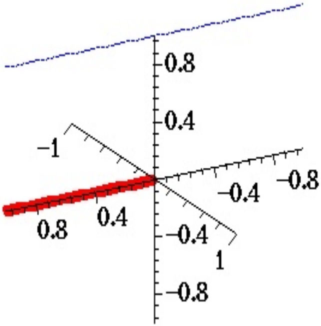
The total field $\mathbf{F}(x_0, y_0, z_0)$ is the sum

$$\mathbf{F}(x_0, y_0, z_0) = \int_0^1 d\mathbf{F}.$$

The Inverse Problem

Suppose we measure only the z component of $\mathbf{F}$, and only at points of the form $x_0 = t, y_0 = 0, z_0 = 1$ (the blue line).



The Inverse Problem

In summary, we measure the quantity

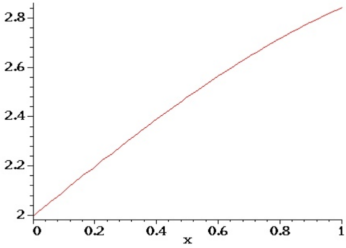
$$\mathbf{F}_z(t, 0, 1) = \int_0^1 \frac{\lambda^*(x)}{((x-t)^2+1)^{3/2}}\,dx$$

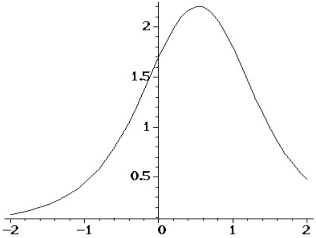
(set $G = 1$ for simplicity) for some range of t, say $-1 \leq t \leq 1$.

Is this enough information to determine $\lambda^*(x)$?

Example

Suppose $\lambda^*(x) = 2 + \sin(x)$. A plot of $\lambda^*(x)$ and $\mathbf{F}_z(t, 0, 1)$:





Is $\lambda^*(x)$ encoded in the graph on the right?

Existence and Uniqueness

It can be shown via Fourier integral transforms that if the function $h(t)$ in

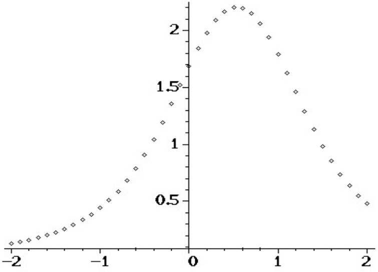
$$\int_0^1 \frac{\lambda^*(x)}{((x-t)^2+1)^{3/2}}\, dx = h(t)$$

is well-behaved on some interval $a < t < b$ then there is a unique solution $\lambda^*(x)$.

We'll just assume away this problem, by supposing our data came from an actual λ^*.

Discrete Data

We wouldn't really have the function $\mathbf{F}_z(t, 0, 1)$, just measurements at discrete points $t = t_1, t_2, \ldots, t_M$:



Finding λ^*: Output Least Squares

Look for the density in the form

$$\lambda(x) = a_0 + a_1 x + a_2 x^2,$$

a quadratic polynomial.

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a quadratic polynomial.

Goal: adjust a_0, a_1, a_2 so this hypothetical density reproduces the real data as closely as possible when we plug λ into the forward model.

Finding λ^*: Output Least Squares

If $\lambda^*(x)$ is the real density, let

$$h_k^* = \int_0^1 \frac{\lambda^*(x)}{((x-t_k)^2+1)^{3/2}}\,dx.$$

(So h_k^* is the real gravitational field at the point $(t_k, 0, 1)$).

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Let

$$h_k(a_0, a_1, a_2) = \int_0^1 \frac{a_0 + a_1 x + a_2 x^2}{((x-t_k)^2+1)^{3/2}}\,dx.$$

We want to adjust a_0, a_1, a_2 to make $h_k(a_0, a_1, a_2) \approx h_k^*$, for all k.

Finding λ: Output Least Squares

Define the objective “fit-to-data” function

$$Q(a_0, a_1, a_2) = \sum_{k=1}^{M} (h_k(a_0, a_1, a_2) - h_k^*)^2.$$

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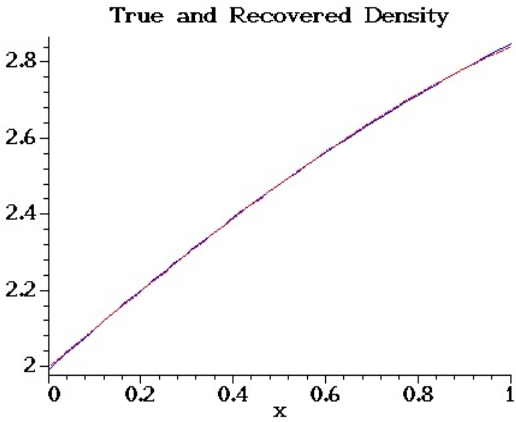
We settle for that a_0, a_1, a_2 that minimizes Q, then take $\lambda(x) = a_0 + a_1 x + a_2 x^2$ as our estimate of the density.

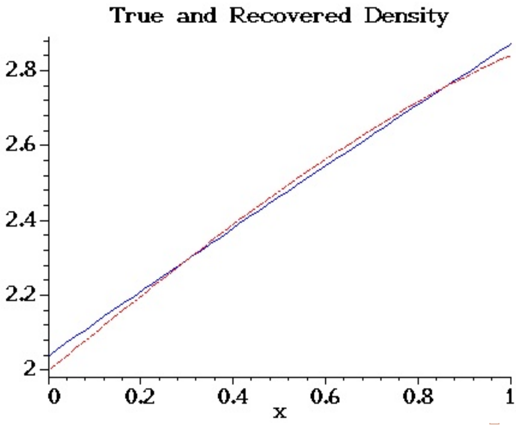
Finding λ: Output Least Squares

The minimum occurs at $a_0 \approx 1.9924$, $a_1 \approx 1.0914$, $a_2 \approx -0.2352$, i.e., $\lambda(x) \approx 1.9924 + 1.0914x - 0.2352x^2$.

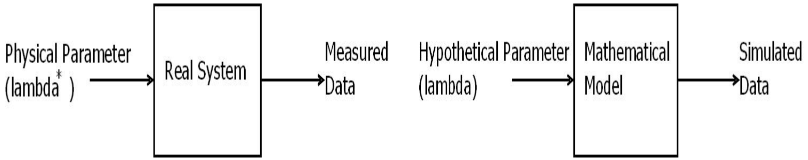


Finding λ: Output Least Squares

If we add noise to the data (uniform in the range -0.005 to 0.005) we obtain



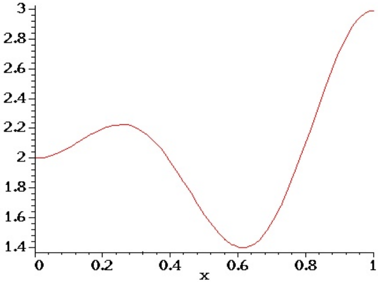
General Least Squares Framework



We adjust λ to minimize $\|\text{measured data} - \text{simulated data}\|^2$.

Gravitational Prospecting Again

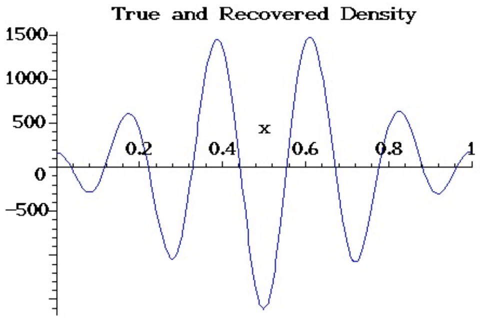
Suppose $\lambda^*(x) = 2 + x\sin(8x)$:



Let's try to fit $\lambda(x) = \sum_{k=0}^{10} a_k \cos(k\pi x)$.

Gravitational Prospecting Again

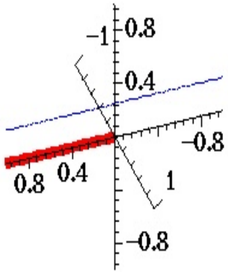
The result (no noise in the data) is



This inverse problem is severely ill-posed!

A Slight Improvement

Moving the gravimeter closer to the bar improves stability.



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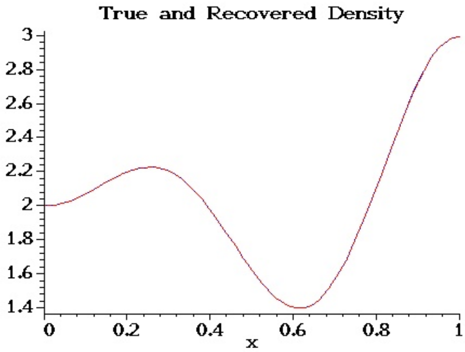
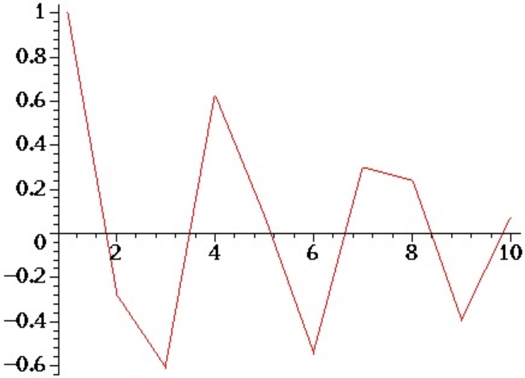


Spring-Mass Inverse Problem

A spring-mass system governed by $x''(t) + cx'(t) + kx(t) = 0$, $x(0) = 1, x'(0) = 0$. We want to find true c^*, k^* from data points $x^*(1), x^*(2), \ldots, x^*(10)$.



Spring-Mass Inverse Problem

Let $x(t)$ be the solution to $x''(t) + cx'(t) + kx(t) = 0$:

$$x(t) = e^{-ct/2}\left(\cos(\omega t) + \frac{c}{2\omega}\sin(\omega t)\right)$$

with $\omega = \sqrt{4k - c^2}/2$.

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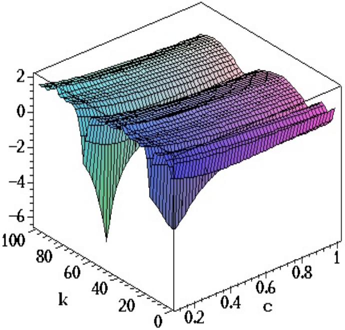
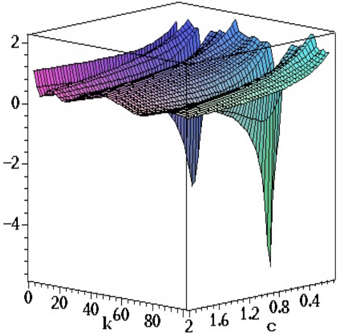
We'll minimize

$$Q(c,k) = \sum_{j=0}^{10}(x(j) - x^*(j))^2.$$

Objective Function Plot

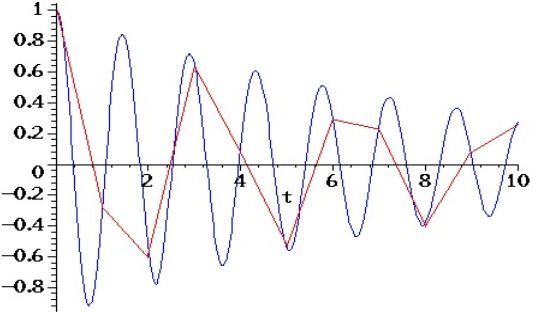
The objective function has many local minima!





The Damping and Spring Estimates

The true damping and spring constants are $c^* = 0.23$ and $k^* = 3.81$. Depending on our starting point for the optimizer we get the correct values, or $(c, k) = (0.23, 18.94)$, $(c, k) = (0.23, 67.6)$, many others. All fit the data well:



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Cons:

- Computationally intensive.
- May generate nonsense if no solution, or solution not unique.
- May get stuck in local min that's not the real solution.
- Doesn't address ill-posedness of the inverse problem.