Introduction to Inverse Problems

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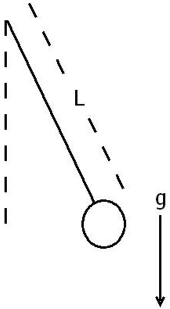
The Pendulum

The period P of a pendulum is $P \approx 2\pi\sqrt{L/g}$.



We can turn this around to find $g \approx 4\pi^2 L/P^2$.

Forward and Inverse Problems

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The Inverse Problem:

- Physical: Given measured data (length) and/or observed behavior (the period) estimate an unknown parameter (gravity).
- Mathematics: Given L and P in $P = 2\pi\sqrt{L/g}$, find g.

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3. Stability: Do small errors in L and P induce small errors in g?
4. Reconstruction: How can we find g from L and P?

Stability

From Calc 3, if $g = 4\pi^2 L/P^2$ then

$$\begin{aligned}\Delta g &= \frac{\partial g}{\partial L}\Delta L + \frac{\partial g}{\partial P}\Delta P \\ &= \frac{4\pi^2}{P^2}\Delta L - \frac{8\pi^2 L}{P^3}\Delta P.\end{aligned}$$

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Divide by $= 4\pi^2 L/P^2$ to find

$$\frac{\Delta g}{g} = \frac{\Delta L}{L} - 2\frac{\Delta P}{P}.$$

This inverse problem is stable.

A More Interesting Problem

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The solution is

$$P(t) = P_0 \exp\left(\int_0^t r(s)\, ds\right).$$

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The solution is just

$$r(t) = P'(t)/P(t).$$

But it's not as simple as it looks...

Estimating the Interest Rate

Suppose we know $P(t)$ at times $t_k = k\Delta t$, $k = 0, 1, 2, \ldots$, rounded to the nearest penny of course. We can estimate

$$P'(t_k) \approx \frac{P(t_{k+1}) - P(t_{k-1})}{2\Delta t}$$

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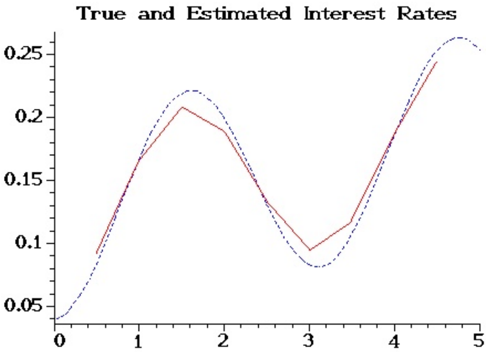
From $r(t) = P'(t)/P(t)$ we get

$$r(t_k) \approx \frac{P(t_{k+1}) - P(t_{k-1})}{2\Delta t P(t_k)}.$$

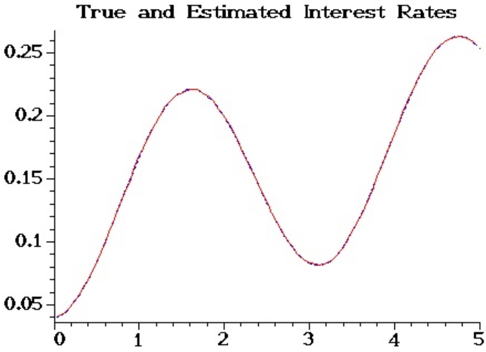
Example

Suppose $r(t) = 0.04(3 - 2\cos(2t) + t/3)$ on $0 \leq t \leq 5$, with $P(0) = 100$. If we use $r(t_k) \approx \frac{P(t_{k+1}) - P(t_{k-1})}{2\Delta t P(t_k)}$ with $\Delta t = 0.5$ the result is



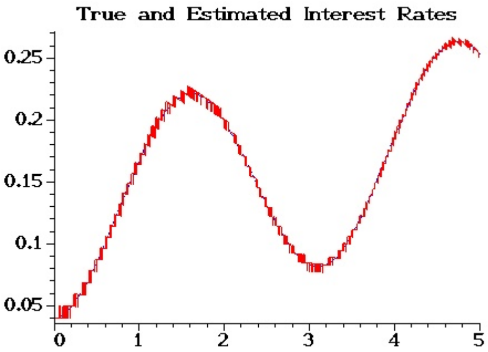
Example

With $\Delta t = 0.05$ the result is better:



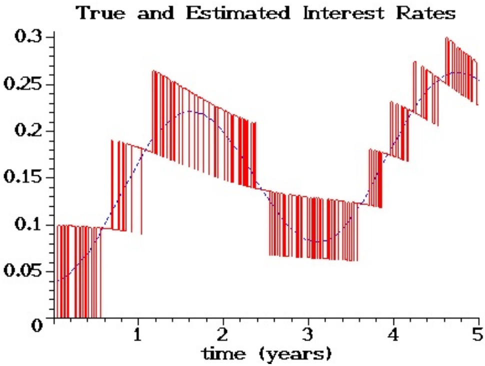
Example

But with $\Delta t = 0.005$ we get



Example

And $\Delta t = 0.0005$ yields



What's Wrong?

We don't really know $P(t_k)$, but $P(t_k)$ rounded to the nearest cent, i.e., we know

$$\tilde{P}(t_k) = P(t_k) + \epsilon_k$$

where $|\epsilon_k| \leq 0.005$ dollars.

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$$\begin{aligned} P'(t_k) &\approx \frac{\tilde{P}(t_{k+1}) - \tilde{P}(t_{k-1})}{2\Delta t} \\ &= \underbrace{\frac{P(t_{k+1}) - P(t_{k-1})}{2\Delta t}}_{\text{better as } \Delta t \to 0} + \underbrace{\frac{\epsilon_{k+1} - \epsilon_{k-1}}{2\Delta t}}_{\text{may blow up as } \Delta t \to 0} . \end{aligned}$$

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- This is typically the case when solving the inverse problem involved estimating derivatives from data.
- Lot's of inverse problems involve differentiating data, especially when the unknown is a function.
- We can stabilize or *regularize* the differentiation of data using a variety of techniques.

Other Basic Issues

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- Linearity—does the unknown depend linearly on the data, or nonlinearly?

Notes URL

www.rose-hulman.edu/~bryan/invprobs.html