Elementary Inversion of the Laplace Transform

Kurt Bryan [1]

Abstract. This paper provides an elementary derivation of a very simple "closed-form" inversion formula for the Laplace Transform.

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1 Introduction

The Laplace transform is a powerful tool in applied mathematics and engineering. Virtually every beginning course in differential equations at the undergraduate level introduces this technique for solving linear differential equations. The Laplace transform is indispensable in certain areas of control theory. Given a function $f(t)$ defined for $0 \leq t < \infty$, the Laplace transform $F(s)$ is defined as

$$F(s) = \int_0^\infty e^{-st} f(t)\, dt, \tag{1}$$

at least for those s for which the integral converges. In practice when one uses the Laplace transform to, for example, solve a differential equation, one has to at some point invert the Laplace transform by finding the function $f(t)$ which corresponds to some specified $F(s)$. The usual technique is to manipulate $F(s)$ algebraically (e.g., with partial fraction decompositions, shifting theorems, etc.) until one can "guess" a function $f(t)$ to which $F(s)$ corresponds. But two questions arise very naturally:

1. Could two different functions $f_1(t)$ and $f_2(t)$ have the same Laplace transform?

2. Is there a constructive or explicit procedure for determining $f(t)$ from $F(s)$?

[1] Department of Mathematics, Rose-Hulman Institute of Technology, Terre Haute IN 47840

Under reasonable restrictions on the functions involved, the answer to both questions is "yes", but the standard techniques used to show this—complex analysis, residue computations, and/or Fourier's integral inversion theorem—are generally outside the scope of an introductory differential equations course.

The purpose of this paper is to show that one can answer both questions above—and give an "explicit" (albeit not necessarily practical) inversion procedure for the Laplace transform—using only very elementary analysis. The main result, Theorem 2.1, was actually proved by E. Post in 1930 [2]. The author of this note "rediscovered" the result while trying to prove results on the non-negativity of a function $f(t)$ from knowledge of its Laplace transform (see Theorem 2.3 below.) The inversion formula does not seem to be well-known, and does not appear in most standard texts on the Laplace Transform.

The actual proof of the inversion formula uses only elementary properties of the Laplace transform as taught in most differential equations courses. Indeed, if one is willing to accept the "graphically" obvious fact stated in Lemma 3.1 below, then the proof of the inversion formula becomes an easy one page exercise.

2 Inverting the Laplace Transform

Let $f(t)$ be a continuous function on the interval $[0, \infty)$ which is of exponential order, that is, for some $b \in \mathbb{R}$

$$\sup_{t>0} \frac{|f(t)|}{e^{bt}} < \infty. \tag{2}$$

In this case the Laplace transform (1) exists for all $s > b$ and is in fact infinitely differentiable with respect to s for $s > b$; see [1], section 19. The following theorem shows that $f(t)$ can be uniquely recovered from $F(s)$.

Theorem 2.1 (Post's Inversion Formula) *A function f which is continuous on $[0, \infty)$ and satisfies the growth condition (2) can be recovered from $F(s)$ as*

$$f(t) = \lim_{k \to \infty} \frac{(-1)^k}{k!} \left(\frac{k}{t}\right)^{k+1} F^{(k)}\left(\frac{k}{t}\right)$$

for $t > 0$, where $F^{(k)}$ denotes the kth derivative of F.

Since Theorem 2.1 yields $f(t)$ in terms of $F(s)$, we immediately obtain

Theorem 2.2 *Let $f(t)$ and $g(t)$ be continuous functions defined for $t \geq 0$ which satisfy (2), with Laplace transforms $F(s)$ and $G(s)$, respectively. If for some constant $c > 0$ we have $F(s) = G(s)$ for all $s > c$ then $f(t) = g(t)$ for all $t > 0$.*

Of course, the main difficulty in using Theorem 2.1 for actually computing the inverse Laplace transform is that repeated symbolic differentiation of F may yield rather unwieldy expressions. However, one can apply the theorem in a few simple cases.

Example 1: Let $f(t) = e^{-at}$. The Laplace transform is easily found to be $F(s) = \frac{1}{s+a}$. It is also simple to verify that $F^{(k)}(s) = k!(-1)^k(s+a)^{-k-1}$. Inserting this expression for $F^{(k)}$ into Theorem 2.1 gives

$$\begin{aligned} f(t) &= \lim_{k\to\infty} \frac{k^{k+1}}{t^{k+1}} \left(a + \frac{k}{t}\right)^{-k-1} \\ &= \lim_{k\to\infty} \left(1 + \frac{at}{k}\right)^{-k-1}. \end{aligned}$$

This last limit is easy to evaluate—take the natural log of both sides and write the resulting indeterminate form as $-\frac{\ln(1+at/k)}{1/(k+1)}$. L'Hopital's rule reveals that the indeterminate form approaches $-at$. The continuity of the natural logarithm immediately shows that $\ln(f(t)) = -at$, so $f(t) = e^{-at}$.

Example 2: Let $f(t) = t^n$, with $n \geq 0$. In this case we have $F(s) = n!s^{-1-n}$. One finds that

$$F^{(k)}(s) = (-1)^k(n+k)!s^{-n-k-1}.$$

Inserting this into Theorem 2.1 yields

$$f(t) = t^n \lim_{k\to\infty} \frac{k^{k+1}(n+k)!}{k!t^{k+1}} \left(\frac{k}{t}\right)^{-n-k-1}.$$

One can use Stirling's formula in the form $\lim_{k\to\infty} \frac{k!}{\sqrt{2\pi k}} k^k e^{-k} = 1$ to obtain, after a bit of simplification,

$$f(t) = t^n e^{-n} \lim_{k\to\infty} \sqrt{1 + \frac{n}{k}} \left(1 + \frac{n}{k}\right)^k \left(1 + \frac{n}{k}\right)^n.$$

As $k \to \infty$ one finds that $\sqrt{1 + \frac{n}{k}} \to 1$, $(1 + \frac{n}{k})^k \to e^n$, and $\left(1 + \frac{n}{k}\right)^n \to 1$. As a result the limit yields $f(t) = t^n$.

Example 3: One can use the inversion formula to approximately invert the Laplace transform for more complicated functions. For example, let

$$f(t) = tH(1-t) + H(t-1) - \frac{1}{2}H(t-2)$$

where $H(t)$ is the Heaviside function ($H(t) = 0$ for $t < 0$, $H(t) = 1$ for $t \geq 0$.) We can compute the Laplace transform $F(s)$ of f to find $F(s) =$

$(1-e^{-s})/s^2 - e^{-2s}/(2s)$. Now apply the inversion formula but replace the limit in k with a finite but reasonably large value of k—say $k = 50$. We have to differentiate $F(s)$ symbolically k times, but a computer algebra system such as Maple or Mathematica makes this tractable. A plot of the successively better reconstructions for $k = 5, 20$, and 50 are shown below:

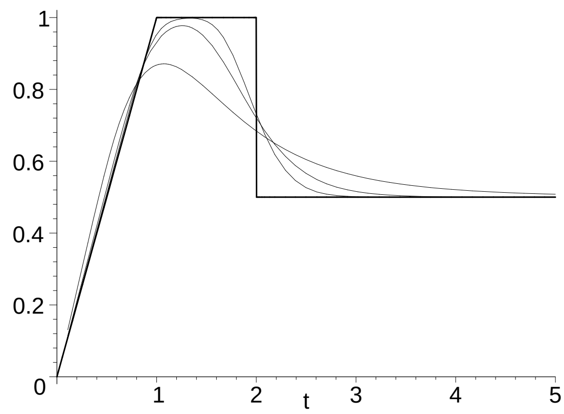

Figure 1: Reconstruction of $f(t)$ for $k = 5, 20, 50$.

Example 4: Identifying Non-negative Functions

Suppose $f(t)$ is a non-negative function which satisfies the bound (2). Then the integrand in the definition of the Laplace transform (1) is non-negative and as a result $F(s) \geq 0$ for all $s > b$. Differentiating (1) repeatedly with respect to s yields

$$F^{(k)}(s) = (-1)^k \int_0^\infty t^k e^{-st} f(t)\, dt.$$

Again, non-negativity of the integrand shows that $F^{(k)}(s)$ alternates sign with respect to k. Thus if $f(t)$ is non-negative then we have $(-1)^k F^{(k)}(s) \geq 0$ for all k and $s > b$.

From Theorem 2.1, however, we see that the converse is also true! If $(-1)^k F^{(k)}(s) \geq 0$ holds for each k then the quantity under the limit on the right side of the expression displayed in Theorem 2.1 is non-negative. As a result, the limit $f(t)$ is necessarily non-negative and we have

Theorem 2.3 *A function f satisfying the bound (2) with Laplace transform $F(s)$ is non-negative if and only if*

$$(-1)^k F^{(k)}(s) \geq 0$$

for all $k \geq 0$ and all $s > b$.

3 Proof of the Inversion Formula

In preparation for the proof of Theorem 2.1, let us define the sequence of functions $g_k(t)$ for $t \geq 0$ and $k \in \mathbf{Z}^+$ as

$$g_k(t) = \frac{k^{k+1}}{k!} t^k e^{-kt}. \tag{3}$$

It is an easy exercise in integration by parts to verify that

$$\int_0^\infty g_k(t)\, dt = 1 \tag{4}$$

for all $k > 0$. Also, g_k has a unique maximum at $t = 1$, with $g_k(1) = \frac{k^{k+1}e^{-k}}{k!}$. Finally, we note that since $k! > k^k e^{-k}$ for all positive integers k (this is easy to prove; see [4], chapter 5, exercises 13 and 14) we have the inequality

$$g_k(t) \leq k t^k e^{k(1-t)} \tag{5}$$

for all $t \geq 0$ and positive k.

Below we plot $g_k(t)$ for several values of k.

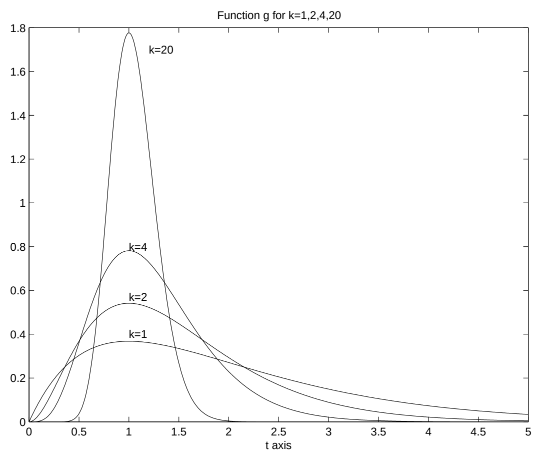

Figure 2: Function $g_k(t)$ for $k = 1, 2, 5, 20$.

It is apparent that as k increases, $g_k(t)$ has a sharper peak and is "localized" near $t = 1$. Put another way, $g_k(t)$ approximates the delta function $\delta(t-1)$. The following lemma quantifies this statement.

Lemma 3.1 *For any continuous function $\phi(t)$ defined for $t \geq 0$ which satisfies the bound (2) we have*

$$\lim_{k\to\infty} \int_0^\infty g_k(t)\phi(t)\, dt = \phi(1).$$

We defer the proof of Lemma 3.1 to section 4. With Lemma 3.1 in hand, however, it is relatively easy to prove Theorem 2.1.

Proof of Theorem 2.1: Given a function $\phi(t)$, let us denote its Laplace transform as a function of s by $\mathcal{L}[\phi](s)$ and set $F(s) = \mathcal{L}[f](s)$. We recall some elementary properties of the Laplace Transform (see [1] for proofs):

1. $\dfrac{d^kF(s)}{ds^k} = \mathcal{L}[(-1)^k t^k f(t)](s)$
2. $F(s+c) = \mathcal{L}[e^{-ct}f(t)](s)$
3. $\dfrac{1}{c}F(s/c) = \mathcal{L}[f(ct)](s)$

For $t_0 > 0$ define the function $\tilde{f}(t) = f(t_0 t)$ and note that by property (3) above we have

$$\mathcal{L}[\tilde{f}](s) = \mathcal{L}[f(t_0 t)](s) = \frac{1}{t_0}F(s/t_0).$$

It immediately follows that for all sufficiently large s

$$\frac{d^k}{ds^k}\left(\mathcal{L}[\tilde{f}]\right)(s) = \frac{1}{t_0^{k+1}}F^{(k)}(s/t_0). \tag{6}$$

Let $\phi(t) = e^{-st}\tilde{f}(t)$. If f satisfies the growth condition (2) then so does ϕ (with b replaced by $b - s$.) From Lemma 3.1 and the definition of $g_k(t)$ we have

$$\begin{aligned} \phi(1) = e^{-s}\tilde{f}(1) = e^{-s}f(t_0) &= \lim_{k\to\infty}\frac{k^{k+1}}{k!}\int_0^\infty e^{-st}t^k e^{-kt}\tilde{f}(t)\,dt \\ &= \lim_{k\to\infty}\frac{k^{k+1}}{k!}\mathcal{L}[t^k e^{-kt}\tilde{f}(t)](s). \end{aligned} \tag{7}$$

Using the above properties (1) and (2) of the Laplace Transform, equation (6), and the definition of $\tilde{f}$ we have

$$\begin{aligned} \mathcal{L}[t^k e^{-kt}\tilde{f}(t)](s) &= (-1)^k\frac{d^k}{ds^k}(\mathcal{L}[e^{-kt}\tilde{f}(t)])(s) \\ &= (-1)^k\frac{d^k}{ds^k}(\mathcal{L}[\tilde{f}])(s+k) \\ &= (-1)^k\frac{1}{t_0^{k+1}}\frac{d^k}{ds^k}(\mathcal{L}[f])\left(\frac{s+k}{t_0}\right) \\ &= (-1)^k\frac{1}{t_0^{k+1}}F^{(k)}\left(\frac{s+k}{t_0}\right). \end{aligned} \tag{8}$$

Equations (7) and (8), with $f(t_0) = e^s\phi(1)$, now yield

$$f(t_0) = e^s \lim_{k\to\infty} \frac{(-1)^k}{k!} \left(\frac{k}{t_0}\right)^{k+1} F^{(k)}\left(\frac{s+k}{t_0}\right) \tag{9}$$

for any s. The statement in Theorem 2.1 is actually just the special case $s = 0$.
■

4 The Proof of Lemma 3.1

We first establish the following technical lemma.

Lemma 4.1 *For any δ with $0 < \delta < 1$ and any real b we have*

$$\lim_{k\to\infty} \int_0^{1-\delta} g_k(t)e^{bt}\,dt = 0, \tag{10}$$

$$\lim_{k\to\infty} \int_{1+\delta}^{\infty} g_k(t)e^{bt}\,dt = 0. \tag{11}$$

Proof: To prove (10), first define the function

$$h(t) = te^{1-t}.$$

The function $h(t)$ is positive for $t > 0$, has a maximum value of 1 at $t = 1$, and $h(t) < 1$ for all $t \neq 1$. Also, it is easy to check that $h'(t) > 0$ for all $t \in (0, 1)$, so h is strictly increasing on $(0, 1)$. It follows that

$$h(t) \le h(1-\delta) < 1$$

for all t with $0 \le t \le 1-\delta$. Note that by inequality (5) we have $g_k(t) \le k(h(t))^k$. Let $m = \max(1, e^{b(1+\delta)})$. It follows that

$$\begin{aligned} 0 < \int_0^{1-\delta} g_k(t)e^{bt}\,dt &\le k\int_0^{1-\delta} (h(t))^k e^{bt}\,dt \\ &< mk\int_0^{1-\delta} (h(1-\delta))^k\,dt \\ &= mk(1-\delta)(h(1-\delta))^k. \end{aligned} \tag{12}$$

Since $0 < h(1-\delta) < 1$, it is clear that the term on the right in (12) approaches zero as $k \to \infty$, and so equation (10) must hold.

To prove equation (11), again let $h(t) = te^{1-t}$. For any a with $0 < a < 1$ and all $t > \frac{1}{1-a}$ we have the inequality

$$h(t) < \frac{1}{1-a}e^{-at}. \tag{13}$$

This can easily be shown by examining the function $\frac{h(t)}{e^{-at}}$. This function has a unique maximum for t in $(1, \infty)$ at $t = \frac{1}{1-a}$, with maximum value $\frac{1}{1-a}$, which immediately implies the inequality (13).

From $g_k(t) \leq k(h(t))^k$ and inequality (13) we obtain

$$g_k(t)e^{bt} < k\left(\frac{1}{1-a}\right)^k e^{(b-ka)t} \tag{14}$$

for all $t > \frac{1}{1-a}$. Let $a_0 = \frac{\delta}{1+\delta}$. Choosing $a < a_0$ gives $0 < a < 1$ and, by simple algebra, $\frac{1}{1-a} < 1 + \delta$. By inequality (14)

$$\begin{aligned} \int_{1+\delta}^{\infty} g_k(t)e^{bt}\,dt &< k\left(\frac{1}{1-a}\right)^k \int_{1+\delta}^{\infty} e^{(b-ka)t}\,dt \\ &= k\left(\frac{1}{1-a}\right)^k \left(\frac{e^{(b-ka)(1+\delta)}}{ka-b}\right) \\ &= \frac{e^{b(1+\delta)}}{a}\left(\frac{e^{-a(1+\delta)}}{1-a}\right)^k . \end{aligned} \tag{15}$$

provided of course that k is large enough so that the integrals converge.

Claim: For any $\delta > 0$ we have, for all sufficiently small positive a, that

$$\frac{e^{-a(1+\delta)}}{1-a} < 1.$$

Proof: By simple algebra the inequality above is equivalent to

$$1 + \delta > \frac{-\ln(1-a)}{a}. \tag{16}$$

It's easy to check that the function $\phi(a) = \frac{-\ln(1-a)}{a}$ satisfies $\lim_{a\to 0}\phi(a) = 1$, $\phi'(a) > 0$ for $0 < a < 1$, and $\lim_{a\to 1}\phi(a) = \infty$. It follows that for $\delta > 0$ the equation

$$\phi(a) = 1 + \delta$$

has a unique solution $a = a_1$. If a is chosen so that $a < a_1$ then inequality (16) will hold, and this proves the claim.

To finish the proof of Lemma 4.1 we choose $a < \min(a_0, a_1)$, and let $d = \frac{e^{-a(1+\delta)}}{1-a} < 1$. Inequality (15) yields

$$0 < \int_{1+\delta}^{\infty} g_k(t)e^{bt}\,dt < \frac{e^{b(1+\delta)}}{a}d^k$$

and since $0 < d < 1$ the right hand side clearly tends to zero as $k \to \infty$, which proves equation (11) and finishes the proof of Lemma 4.1.■

An immediate consequence of Lemma 4.1 and equation (4) is that

$$\lim_{k\to\infty} \int_{1-\delta}^{1+\delta} g_k(t)\,dt = 1. \tag{17}$$

We can now prove Lemma 3.1.

Proof of Lemma 3.1: Since ϕ is continuous we may, for any $\epsilon > 0$, choose some δ so that $|\phi(t) - \phi(1)| < \epsilon$ for all t with $1 - \delta < t < 1 + \delta$. We then have

$$\begin{aligned} \int_0^\infty g_k(t)\phi(t)\,dt - \int_{1-\delta}^{1+\delta} g_k(t)\phi(1)\,dt &= \int_{1-\delta}^{1+\delta} g_k(t)(\phi(t) - \phi(1))\,dt \\ &+ \int_0^{1-\delta} g_k(t)\phi(t)\,dt + \int_{1+\delta}^\infty g_k(t)\phi(t)\,dt \end{aligned}$$

so that

$$\begin{aligned} \left|\int_0^\infty g_k(t)\phi(t)\,dt - \int_{1-\delta}^{1+\delta} g_k(t)\phi(1)\,dt\right| &\leq \epsilon \int_{1-\delta}^{1+\delta} g_k(t)\,dt \\ &+ \left|\int_0^{1-\delta} g_k(t)\phi(t)\,dt\right| + \left|\int_{1+\delta}^\infty g_k(t)\phi(t)\,dt\right|. \end{aligned}$$

From Lemma 4.1 and the fact that ϕ is of exponential order we find that the last two integrals on the right above approach zero as $k \to \infty$, and since ϵ is arbitrary we conclude that

$$\lim_{k\to\infty} \int_0^\infty g_k(t)\phi(t)\,dt = \lim_{k\to\infty} \int_{1-\delta}^{1+\delta} g_k(t)\phi(1)\,dt = \phi(1)$$

by equation (17), which proves the claim. ■

5 Conclusion

As noted, Theorem 2.1 does not provide a very practical means for inverting the Laplace Transform in general, since repeated differentiation of $F(s)$ generally leads to unwieldy expressions. Nonetheless, it does illustrate why the inversion of the Laplace Transform is so ill-posed, for the inversion formula requires derivatives of arbitrarily high order, impossible to compute from real data.

It would be interesting to consider what quantity the inversion formula would recover when applied to a function with jump discontinuities. Specifically, let f be continuous from the left and right at a point x, but have a jump discontinuity at x, with $f(x^+)$ and $f(x^-)$ the limits from the right and left, respectively. Based on Figure 2 (and simple numerical experiments) one would expect that the inversion formula should recover $(f(x^+)+f(x^-))/2$, but the author has not seen nor written out a proof of this.

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