Let $E$ be the elliptic curve

$$E : y^2 + a_1 xy + a_3 y = x^3 + a_2 x^2 + a_4 x + a_6.$$ 

Let $P = (x_0, y_0) \in E$. then

$$-P = (x_0, -y_0 - a_1 x_0 - a_3).$$

Let $P_1 + P_2 = P_3$, with $P_1 = (x_1, y_1) \in E$. If $x_1 = x_2$ and $y_1 + y_2 + a x_2 + a_3 = 0$, the $P_1 + P_2 = O$.

Otherwise,

- If $x_1 \neq x_2$: $m = \frac{y_2 - y_1}{x_2 - x_1}$, $\nu = \frac{y_1^2 - y_2^2}{x_2 - x_1}$
- If $x_1 = x_2$: $m = \frac{3x_1^2 + 2a_2 x + a_4}{2y_1 + a_1 x + a_3}$, $\nu = \frac{x_1^2 + a_1 x_1 + 2a_6 - a_3 y_1}{2y_1 + a_1 x + a_3}$

Then $P_3 = P_1 + P_2$ is given by

- $x_3 = m^2 + a_1 m - a_2 - x_1 - x_2$
- $y_3 = 9(m + a_1)x_3 - \nu - a_3$.

When we are looking simply at the x-coordinate, we have

$$x(P_1 + P_2) = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)^2 + a_1 \left(\frac{y_2 - y_1}{x_2 - x_1}\right) - a_2 - x_1 - x_2,$$

and

$$x([2]P) = \frac{x^4 - b_4 x^2 - 2b_6 x - b_8}{4x^3 + b_2 x^2 + 2b_4 x + b_6}.$$
The general Weierstrass equation in projective coordinates \((X,Y,Z)\), given by

\[ Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 \]

may be written using the non-homogeneous coordinates \(x = X/Z\), \(y = Y/Z\) as

\[ E : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6. \]

If the characteristic of the field is not 2, we may replace \(y\) by \((y - a_1x - a_3)/2\) to get an equation of the form

\[ E : y^2 = 4x^3 + b_2x^2 + 2b_4x + b_6, \]

where

\[
\begin{align*}
b_2 &= a_1^2 + 4a_2, \\
b_4 &= 2a_4 + a_1a_3, \\
b_6 &= a_3^2 + 4a_6.
\end{align*}
\]

Other useful quantities are

\[
\begin{align*}
b_8 &= a_1^2a_6 + 4a_2a + 6 - a_1a_3a_4 + a_2a_3^2 - a_1^2, \\
c_4 &= b_2^2 - 24b_4, \\
c_6 &= b_2^3 + 36b_2b_4 - 216b_6, \\
\Delta &= -b_2^2b_8 - 8b_4^3 - 27b_6^2 + 9b_2b_4b_6, \\
j &= c_4^3/\Delta.
\end{align*}
\]

These are related by \(4b_8 = b_2b_6 - b_1^2\) and \(1728\Delta = c_4^3 - c_6^2\).

If the characteristic is neither 2 nor 3, then \((x, y)\) may be replaced by \(\left(\frac{x - 3b_2}{36}, \frac{y}{216}\right)\) to yield the simpler equation

\[ E : y^2 = x^3 - 27c_4x - 54c_6. \]

\(\Delta\) is the discriminant of the Weierstrass equation, and \(j\) is called the \(j\)-invariant of the elliptic curve.

If \(E : y^2 = x^3 + Ax + B\), then \(\Delta = -16(4A^3 + 27B^2)\), and \(j = 1728(4A)^3/\Delta\).

The only change of variables preserving this form of the curve is \(x = u^2x', y = u^3y'\). This gives \(u^4A' = A, u^6B' = B,\) and \(u^{12}\Delta' = \Delta\).