1 (A) What are the roots modulo 7 of the polynomial \( P(x) = x^7 - x \)?

(B) Determine the roots of \( f(x) = x^9 + x^7 - x^3 + x - 6 \) modulo 7 by applying the division algorithm using the polynomial \( P(x) = x^7 - x \) to get polynomials \( q(x), r(x) \) satisfying \( f(x) = P(x)q(x) + r(x) \), where \( r(x) \) has degree less than \( P(x) \).

(C) Factor the polynomial \( f(x) = x^9 + x^7 - x^3 + x - 6 \) modulo 7 into linear terms and a non-linear polynomial, \( NL(x) \).

Can you show that the non-linear polynomial has no roots modulo 7? Start by looking at \( NL(x) - 2 \).

2 The polynomial \( f(x) = 2x^3 - 2x^2 - 2x + 2 \) has roots modulo \( 10^{20} - 1 \) at \( x = 1, 10, -10 \). Show that if \( 10^{20} - 1 \) were prime, then \( f(x) \) has no other roots, by considering \( f(x) - k \cdot (x - 1)(x - 10^{10})(x + 10^{10}) \) modulo \( 10^{20} - 1 \) for an appropriate value of \( k \).

Since \( 10^{20} - 1 \) factors, it is possible to find other roots to the polynomial by using the Chinese Remainder Theorem. What are the factors of \( 10^{20} - 1 \)? What does this imply for the number of roots of \( f(x) \)?

3 (A) Let \( p \) be an odd prime. Determine the values of the polynomial \( 1 - (x - a)^{p-1} \) modulo \( p \) for all residues \( x \).

(B) Construct a polynomial \( f(x) \) so that \( f(5) \equiv 7 \pmod{11} \) and \( f(x) \equiv 0 \pmod{11} \) if \( x \not\equiv 5 \pmod{11} \).

(C) Construct a polynomial \( f(x) \) so that \( f(x) = \begin{cases} 7 \pmod{11} & \text{if } x \equiv 5 \pmod{11} \\ 3 \pmod{11} & \text{if } x \equiv 7 \pmod{11} \\ 0 \pmod{11} & \text{otherwise} \end{cases} \)

Theorem 2.29 in the book (page 94) states that the congruence \( f(x) \equiv 0 \pmod{p(x)} \) of degree \( n \), with leading coefficient \( a_n = 1 \), has \( n \) solutions if and only if \( f(x) \) is a factor of \( x^p - x \) modulo \( p \).

4 (A) Use Theorem 2.29 to prove that if \( d \mid (p-1) \) then \( x^d \equiv 1 \pmod{p} \) has exactly \( d \) solutions.

(B) The corollary that you just proved implies that \( x^3 \equiv 1 \pmod{p} \) has exactly three solutions modulo \( p \) when \( p \equiv 1 \pmod{3} \). \( p = 195, 019, 441 \) is a prime congruent to 1 modulo 3. \( 161051^3 \equiv 1 \pmod{p} \). Find the other two roots. (Hint: factor the polynomial!)