Numerical Integration of a Single ODE

In the last lecture, we learned how to perform numerical integration to find out the area under an arbitrary curve. Today, we will extend our method further to obtain a numerical solution to a single ordinary differential equation. Don’t be scared by the terminology here ☺ The idea is basically the same as finding the area under a prescribed curve. If you finish the in-class examples and homework from last lecture, you will be doing fine today.

**Theory**

Given an ordinary differential equation:

$$\frac{dy}{dx} = f(x, y) \quad \text{with} \quad y(x = x_0) = y_0$$

The objective of the problem is to solve for the function $y(x)$ which is governed by the above ordinary differential equation and the initial condition. We will be using a marching technique to find out the function $y(x)$. The concept of marching can be simply phrased as: your present conditions fully determine your immediate future. It can be understood as follows.

You may think of the slope of the tangent to the unknown function $y(x)$ at $x = x_0$ is given by

$$\frac{dy}{dx} = f(x_0, y_0) \quad \text{at} \quad x = x_0$$

and represented by Line Segment 1 in the figure below.

![Diagram of Line Segment 1](image)

By knowing the starting value of $y$ at $x = x_0$, the immediate future value of the function $y$ at $x = x_1$ can be obtained by a crude linear approximation:

$$y_1 = y_0 + \left( \frac{dy}{dx} \right)_{(x_0, y_0)}(x_1 - x_0)$$

In this way, the solution at $x = x_1$ can be interpreted as a modification to the starting value of $y$ at $x = x_0$ by the amount

$$\left( \frac{dy}{dx} \right)_{(x_0, y_0)}(x_1 - x_0)$$
After obtaining the functional value at $x = x_1$, the above procedure can be repeated indefinitely to find the functional values at $x = x_2, x_3, x_4, \ldots$. This is the concept of time marching!

The simple marching method introduced above is called the Explicit Euler method which is based on a linear approximation. It is not the most accurate method but it captures the fundamentals of time marching. As expected, the accuracy of the solution can be improved by reducing the step size of time marching. In other words, reducing the region where the linear approximation is used.

Due to the repetitive nature of the marching procedure, the for loop structure is perfect for its implementation.

**Example:**
For the remaining part of this lecture, your task is to write a Matlab program to implement the Explicit Euler method to solve the following differential equation up to $x = 5$:

$$\frac{dy}{dx} = x + y \quad \text{with} \quad y(x = 0) = 1$$

The exact solution to the above equation is $y(x) = 2 \exp(x) - x - 1$ Compare the numerical and exact solution by plotting them on the same graph. Also, plot the error versus $x$ to observe the error propagation behavior.

Fill in the following table and comment on the effect of step size on the error in the final solution at $x = 5$.

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<th>Time step</th>
<th>$y(x = 5)$</th>
<th>Error at $x = 5$</th>
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