Numerical Integration

Theory
The mathematical concept known as integration is widely used in all kinds of engineering applications. Suppose that the function $f(x)$ has been defined on an interval of the real line $[a, b]$. The definite integral of $f(x)$ on this interval may be written

$$Q = \int_{a}^{b} f(x) \, dx$$

The graphical interpretation is very simple. $Q$ is just the area under the curve described by $f(x)$ as we plot it between the endpoints $a$ and $b$.

Please note that the area below the curve is negative, and the area above it is positive. In the sketch above observe that $Q = Q_1 + Q_2 + Q_3$. The concept that you have to grasp is called the Fundamental Theorem of the Calculus.

Say we have a function $F(x)$ whose derivative happens to be $f(x)$. Then, according to the Fundamental Theorem, $Q = F(b) - F(a)$. This implies that integration starts with a derivative and tries to find the function it came from. If you do so then the theorem gives you a mathematically exact result. In many cases, however, the mathematically exact result is very hard or even impossible to obtain. We settle for an approximation. MATLAB is ideally suited for calculating approximations to $Q$.

How? Well, the idea is based on a “divide and conquer” approach. We can divide the integral into “panels” whose areas can be added together to form an approximation to $Q$. Let’s consider for a moment the simple function: $f(x) = 4x^3 + 1$. Let’s integrate this function from 0 to 1. For starters, and as a review, write the MATLAB commands necessary to plot this function on the interval of interest. Use 101 points spaced at intervals of 0.01 between 0 and 1.

$$\begin{align*}
\text{>> } & \quad \\
\text{>> } & \quad \\
\text{>> } & \quad \\
\text{>> } & \quad \\
\text{>> } & \quad \\
\end{align*}$$
Here is what you should see.

Let’s divide the area under the curve into trapezoidal shaped panels. We’ll start with two panels. Divide the area under the curve into two trapezoids which meet at \( x = 0.5 \). The areas are:

Left Trapezoid_____________________. Right Trapezoid_____________________. Total________________

Let’s do it in MATLAB

```matlab
>> x = [ 0 0.5 1];
>> Q = 0;
>> Q = Q + 0.5 .* ( f(1) + f(2) )./2;
>> Q = Q + 0.5 .* ( f(2) + f(3) )./2;
```

The answer here should look familiar. Also note the pattern of the formula. It is ideal for looping. So your next job is to program a ‘.m’ file which can be directed to calculate \( Q \) with any number of panels. It should have a for loop. (Hint: please use the ‘;’ ending on commands inside the loop when the number of panels is large.) Use your program to fill out the following table. The exact solution for \( Q \) is 2.

<table>
<thead>
<tr>
<th>Number of Panels</th>
<th>Panel Size</th>
<th>Computed Value for ( Q )</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>0.50</td>
<td>2.25</td>
<td>0.25 x 10^0</td>
</tr>
<tr>
<td>4</td>
<td>0.25</td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>0.10</td>
<td></td>
<td></td>
</tr>
<tr>
<td>100</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1000</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10000</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here’s a question to think about. As the number of panels grows large, how does the error in \( Q \) depend on the width of the panel which is also called the step size?
Example:

A large storage tank must be drained with gravity flow. A simple mathematical model describes the relationship between depth of fluid left in the tank, $h$, and time, $t$. Assume the depth starts at 8 meters. The time rate of change of depth is given by

$$\frac{dh}{dt} = -0.0005\sqrt{h} \quad h(0) = 8$$

The minus sign tells us that it’s draining. This can be written in another way, with the variables separated.

$$\frac{dh}{-0.0005\sqrt{h}} = dt$$

To calculate the time $T$ to drain the tank down to a basically an empty depth of 0.01 m, we do some minor algebra and integrate $h$ between 0.01 and 8. Time will be in seconds and depth will be in meters.

$$T = \int_{0.01}^{8} \frac{2000}{\sqrt{h}} \, dh$$

Reprogram the ‘.m’ file that you were working on earlier to handle this problem. Integrate to find the total time to drain the tank. Use enough steps to convince yourself that your error is less than 1.0%.

Time to drain________________________.   Number of panels or steps used_________________________.