Multi-DOF systems

Today’s Objectives:

Students will be able to:

a) Find natural frequencies and modes of MDOF systems

b) Have a better understanding of the structural eigenvalue problem.

Note: Bring your laptop to class tomorrow!
Multi-degree of freedom systems

Linear EOM for a MDOF system:

$$\begin{bmatrix} M \end{bmatrix}\ddot{x} + \begin{bmatrix} C \end{bmatrix}\dot{x} + \begin{bmatrix} K \end{bmatrix}x = \{F\}$$

Fig. 2  Simple full car ride model (7-DOF model)

Finite element model of a pressure vessel

Rose-Hulman Institute of Technology
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Solution Procedure for MDOF

1. Find the EOM (M and K)

\[ [M] \ddot{x} + [K] x = 0 \]

2. Assume simple harmonic motion to obtain:

\[ (-\omega^2[M] + [K])x = 0 \Rightarrow \omega^2[M]x = [K]x \]

3. Non-dimensionalize the equations if necessary (for example if using Matlab)

\[ \bar{\omega}^2[\bar{M}]x = [\bar{K}]x \]

4. Find the frequencies and modes (eigenvalues and eigenvectors).

\[ [\text{evec,eval}] = \text{eig}(k,m) \]

Note: If using a program that cannot solve the generalized eigenvalue problem then you will need to write the problem as a standard eigenvalue problem by multiplying by the inverse of \([M]\) or \([K]\)
Let’s talk more about eigenvalue problems (EVP)

The standard eigenvalue problem is usually written as:

\[
[A] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}
\]

This can be written as

\[
([A] - \lambda[I]) \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

or

\[
\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots \\ a_{21} & a_{22} - \lambda & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]

For a non-trivial solution we require

\[
[A] - \lambda[I] = 0
\]

\[
\lambda^n + C_1 \lambda^{n-1} + C_2 \lambda^{n-2} + \cdots + C_{n-1} \lambda + C_n
\]

The roots of this equation are called the eigenvalues of \([A]\)

\[
\lambda_1 \leq \lambda_2 \leq \lambda_3 \cdots \leq \lambda_n
\]
Corresponding to each eigenvalue is an eigenvector

For each eigenvalue there is an eigenvector \( \{x\}_j \) such that

\[
\begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]

Notes:

- Eigenvectors are not unique. They can only be determined to an arbitrary multiplicative constant
- We only have \( n-1 \) independent equations. One method to determine the elements of \( \{x\}_j \) is to let \( x_{1j} = 0 \) and solve the remaining \( n-1 \) equations

We will often “normalize” the equations. One way to do this is:

If we do this, we say the eigenvectors are normalized to have unit magnitudes
For structural systems we get a generalized (or structural) eigenvalue problem

The structural EVP is:

\[ \omega^2 [M] \{X\} = [K] \{X\} \]

This can be written as

\[ (-\omega^2 [M] + [K])\{X\} = \{0\} \]

For a non-trivial solution we require the determinate = 0

\[ |(-\omega^2 [M] + [K])| = 0 \]

To get the eigenvectors we substitute the eigenvalues back in

\[ (-\omega_i^2 [M] + [K])\{X\}_i = \{0\} \]
Determine the natural frequencies and modes for the system shown below.

Using Lagrange’s Equations we get:

\[
\begin{bmatrix}
m & 0 & 0 \\
0 & m & 0 \\
0 & 0 & 2m \\
\end{bmatrix}\begin{bmatrix}
\ddot{x}_1 \\
\ddot{x}_2 \\
\ddot{x}_3 \\
\end{bmatrix} + \begin{bmatrix}
2k & -k & 0 \\
-k & 3k & -2k \\
0 & -2k & 2k \\
\end{bmatrix}\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix}
\]
Let’s talk about **orthogonality** and **normalization** of eigenvectors (i.e. modes) in some more detail.

For $\omega_i$:
$$\omega_i^2 [M] \{X\}_i = [K] \{X\}_i$$  \hspace{1cm} (1)

For $\omega_j$:
$$\omega_j^2 [M] \{X\}_j = [K] \{X\}_j$$  \hspace{1cm} (2)

Multiply (1) by $\{X\}_j^T$ giving:
$$\omega_i^2 \{X\}_j^T [M] \{X\}_i = \{X\}_j^T [K] \{X\}_i$$  \hspace{1cm} (3)

Multiply (2) by $\{X\}_i^T$ giving:
$$\omega_j^2 \{X\}_i^T [M] \{X\}_j = \{X\}_i^T [K] \{X\}_j$$  \hspace{1cm} (4)

For a symmetric matrix $\{X\}_i^T [A] \{X\}_j = \{X\}_j^T [A] \{X\}_i$

So we can write (3) and (4) as

$$\omega_i^2 \{X\}_j^T [M] \{X\}_i = \{X\}_j^T [K] \{X\}_i$$  \hspace{1cm} (3)

$$\omega_j^2 \{X\}_i^T [M] \{X\}_j = \{X\}_i^T [K] \{X\}_j$$  \hspace{1cm} (4)
Orthogonality (cont.)

From the previous page:

\[
\omega_i^2 \{X\}_j^T [M] \{X\}_i = \{X\}_j^T [K] \{X\}_i
\]  
(3)

\[
\omega_j^2 \{X\}_j^T [M] \{X\}_i = \{X\}_j^T [K] \{X\}_i
\]  
(4)

Subtracting these equations we get

\[
(\omega_i^2 - \omega_j^2) \{X\}_j^T [M] \{X\}_i = 0
\]

If we have distinct eigenvalues, that is, \( \omega_i \neq \omega_j \) then we get

Substituting this into (3) gives

So, the eigenvectors (or normal modes or natural modes) are said to be orthogonal with respect to the mass and stiffness matrices. Therefore

1. They are linearly independent
2. The form a basis in n-dimensional space
   a. A basis is a set of linearly independent vector
   b. Any vector can be written in terms of the basis vectors
Normalization of the eigenvectors

Let

Then we get

If this is true, the modes are called “mass normalized”

If we define the modal matrix $[\Phi]$ (book uses $[X]$) to be

$$[\Phi] = \begin{bmatrix} \{X\}_1 & \{X\}_2 & \cdots & \{X\}_n \end{bmatrix}$$

Matrix whose columns are the eigenvectors

then

Doing this in Matlab

The current version of Matlab will give mass normalized modes.

\[ [\text{evec}, \text{eval}] = \text{eig}(k, m) \]

If you need to do it manually you would use code like the following (assuming you have “evec” and “eval” as your eigenvalues and eigenvectors):

\[
[\text{eval}, j] = \text{sort}(\text{diag}(\text{eval})) \quad \% \text{this sorts the eigenvalues}
\]
\[
\text{evec} = \text{evec}(\cdot, j) \quad \% \text{this sorts the eigenvectors}
\]

To mass normalize (if not already normalized):

\[
\text{evec} = \text{evec} / \sqrt{\text{diag}(\text{diag}((\text{evec}^\prime \ast m \ast \text{evec})))}
\]