CHAPTER OUTLINE

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LEARNING OBJECTIVES

After careful study of this chapter you should be able to do the following:
1. Determine probabilities from probability density functions.
2. Determine probabilities from cumulative distribution functions and cumulative distribution functions from probability density functions, and the reverse.
3. Calculate means and variances for continuous random variables.
4. Understand the assumptions for each of the continuous probability distributions presented.
5. Select an appropriate continuous probability distribution to calculate probabilities in specific applications.
6. Calculate probabilities, determine means and variances for each of the continuous probability distributions presented.
7. Standardize normal random variables.
8. Use the table for the cumulative distribution function of a standard normal distribution to calculate probabilities.


**CD MATERIAL**

10. Use continuity corrections to improve the normal approximation to those binomial and Poisson distributions.

Answers for most odd numbered exercises are at the end of the book. Answers to exercises whose numbers are surrounded by a box can be accessed in the e-Text by clicking on the box. Complete worked solutions to certain exercises are also available in the e-Text. These are indicated in the Answers to Selected Exercises section by a box around the exercise number. Exercises are also available for some of the text sections that appear on CD only. These exercises may be found within the e-Text immediately following the section they accompany.

### 4.1 CONTINUOUS RANDOM VARIABLES

Previously, we discussed the measurement of the current in a thin copper wire. We noted that the results might differ slightly in day-to-day replications because of small variations in variables that are not controlled in our experiment—changes in ambient temperatures, small impurities in the chemical composition of the wire, current source drifts, and so forth.

Another example is the selection of one part from a day’s production and very accurately measuring a dimensional length. In practice, there can be small variations in the actual measured lengths due to many causes, such as vibrations, temperature fluctuations, operator differences, calibrations, cutting tool wear, bearing wear, and raw material changes. Even the measurement procedure can produce variations in the final results.

In these types of experiments, the measurement of interest—current in a copper wire experiment, length of a machined part—can be represented by a random variable. It is reasonable to model the range of possible values of the random variable by an interval (finite or infinite) of real numbers. For example, for the length of a machined part, our model enables the measurement from the experiment to result in any value within an interval of real numbers. Because the range is any value in an interval, the model provides for any precision in length measurements. However, because the number of possible values of the random variable \( X \) is uncountably infinite, \( X \) has a distinctly different distribution from the discrete random variables studied previously. The range of \( X \) includes all values in an interval of real numbers; that is, the range of \( X \) can be thought of as a continuum.

A number of continuous distributions frequently arise in applications. These distributions are described, and example computations of probabilities, means, and variances are provided in the remaining sections of this chapter.

### 4.2 PROBABILITY DISTRIBUTIONS AND PROBABILITY DENSITY FUNCTIONS

Density functions are commonly used in engineering to describe physical systems. For example, consider the density of a loading on a long, thin beam as shown in Fig. 4-1. For any point \( x \) along the beam, the density can be described by a function (in grams/cm). Intervals with large loadings correspond to large values for the function. The total loading between points \( a \) and \( b \) is determined as the integral of the density function from \( a \) to \( b \). This integral is the area
under the density function over this interval, and it can be loosely interpreted as the sum of all the loadings over this interval.

Similarly, a probability density function $f(x)$ can be used to describe the probability distribution of a continuous random variable $X$. If an interval is likely to contain a value for $X$, its probability is large and it corresponds to large values for $f(x)$. The probability that $X$ is between $a$ and $b$ is determined as the integral of $f(x)$ from $a$ to $b$. See Fig. 4-2.

**Definition**

For a continuous random variable $X$, a probability density function is a function such that

\begin{align*}
(1) \quad f(x) &\geq 0 \\
(2) \quad \int_{-\infty}^{\infty} f(x) \, dx &= 1 \\
(3) \quad P(a \leq X \leq b) &= \int_{a}^{b} f(x) \, dx = \text{area under } f(x) \text{ from } a \text{ to } b \\
&\text{for any } a \text{ and } b
\end{align*}

(4-1)

A probability density function provides a simple description of the probabilities associated with a random variable. As long as $f(x)$ is nonnegative and $\int_{-\infty}^{\infty} f(x) \, dx = 1$, $0 \leq P(a < X < b) \leq 1$ so that the probabilities are properly restricted. A probability density function is zero for $x$ values that cannot occur and it is assumed to be zero wherever it is not specifically defined.

A histogram is an approximation to a probability density function. See Fig. 4-3. For each interval of the histogram, the area of the bar equals the relative frequency (proportion) of the measurements in the interval. The relative frequency is an estimate of the probability that a measurement falls in the interval. Similarly, the area under $f(x)$ over any interval equals the true probability that a measurement falls in the interval.

The important point is that $f(x)$ is used to calculate an area that represents the probability that $X$ assumes a value in $[a, b]$. For the current measurement example, the probability that $X$ results in $[14 \text{ mA}, 15 \text{ mA}]$ is the integral of the probability density function of $X$ over this interval. The probability that $X$ results in $[14.5 \text{ mA}, 14.6 \text{ mA}]$ is the integral of
the same function, \( f(x) \), over the smaller interval. By appropriate choice of the shape of \( f(x) \), we can represent the probabilities associated with any continuous random variable \( X \). The shape of \( f(x) \) determines how the probability that \( X \) assumes a value in \([14.5 \text{ mA}, 14.6 \text{ mA}]\) compares to the probability of any other interval of equal or different length.

For the density function of a loading on a long thin beam, because every point has zero width, the loading at any point is zero. Similarly, for a continuous random variable \( X \) and any value \( x \).

Based on this result, it might appear that our model of a continuous random variable is useless. However, in practice, when a particular current measurement is observed, such as 14.47 milliamperes, this result can be interpreted as the rounded value of a current measurement that is actually in a range such as \( 14.465 \leq x \leq 14.475 \). Therefore, the probability that the rounded value 14.47 is observed as the value for \( X \) is the probability that \( X \) assumes a value in the interval \([14.465, 14.475]\), which is not zero. Similarly, because each point has zero probability, one need not distinguish between inequalities such as \( < \) or \( \leq \) for continuous random variables.

If \( X \) is a continuous random variable, for any \( x_1 \) and \( x_2 \),

\[
P(x_1 \leq X \leq x_2) = P(x_1 < X \leq x_2) = P(x_1 \leq X < x_2) = P(x_1 < X < x_2) \quad (4-2)
\]

**EXAMPLE 4-1**

Let the continuous random variable \( X \) denote the current measured in a thin copper wire in milliamperes. Assume that the range of \( X \) is \([0, 20 \text{ mA}]\), and assume that the probability density function of \( X \) is \( f(x) = 0.05 \) for \( 0 \leq x \leq 20 \). What is the probability that a current measurement is less than 10 milliamperes?

The probability density function is shown in Fig. 4-4. It is assumed that \( f(x) = 0 \) wherever it is not specifically defined. The probability requested is indicated by the shaded area in Fig. 4-4.

\[
P(X < 10) = \int_{0}^{10} f(x) \, dx = \int_{0}^{10} 0.05 \, dx = 0.5
\]
As another example,

\[ P(5 < X < 20) = \int_{5}^{20} f(x) \, dx = 0.75 \]

**EXAMPLE 4-2**  
Let the continuous random variable \( X \) denote the diameter of a hole drilled in a sheet metal component. The target diameter is 12.5 millimeters. Most random disturbances to the process result in larger diameters. Historical data show that the distribution of \( X \) can be modeled by a probability density function \( f(x) = 20e^{-20(x-12.5)}, x \geq 12.5 \).

If a part with a diameter larger than 12.60 millimeters is scrapped, what proportion of parts is scrapped? The density function and the requested probability are shown in Fig. 4-5. A part is scrapped if \( X > 12.60 \). Now,

\[ P(X > 12.60) = \int_{12.6}^{\infty} f(x) \, dx = \left[ 20e^{-20(x-12.5)} \right]_{12.6}^{\infty} = 0.135 \]

What proportion of parts is between 12.5 and 12.6 millimeters? Now,

\[ P(12.5 < X < 12.6) = \int_{12.5}^{12.6} f(x) \, dx = \left[ -e^{-20(x-12.5)} \right]_{12.5}^{12.6} = 0.865 \]

Because the total area under \( f(x) \) equals 1, we can also calculate \( P(12.5 < X < 12.6) = 1 - P(X > 12.6) = 1 - 0.135 = 0.865 \).

**EXERCISES FOR SECTION 4-2**

4-1. Suppose that \( f(x) = e^{-x} \) for \( 0 < x \). Determine the following probabilities:
   (a) \( P(1 < X) \)  \( (b) \ P(1 < X < 2.5) \)
   (c) \( P(X = 3) \)  \( (d) \ P(X < 4) \)
   (e) \( P(3 \leq X) \)

4-2. Suppose that \( f(x) = e^{-x} \) for \( 0 < x \).
   (a) Determine \( x \) such that \( P(x < X) = 0.10 \).
   (b) Determine \( x \) such that \( P(X \leq x) = 0.10 \).

4-3. Suppose that \( f(x) = x/8 \) for \( 3 < x < 5 \). Determine the following probabilities:
   (a) \( P(X < 4) \)  \( (b) \ P(X > 3.5) \)
   (c) \( P(4 < X < 5) \)  \( (d) \ P(X < 4.5) \)
   (e) \( P(X < 3.5 \text{ or } X > 4.5) \)

4-4. Suppose that \( f(x) = e^{-(x-4)} \) for \( 4 < x \). Determine the following probabilities:
   (a) \( P(1 < X) \)  \( (b) \ P(2 \leq X < 5) \)
4-3 CUMULATIVE DISTRIBUTION FUNCTIONS

An alternative method to describe the distribution of a discrete random variable can also be used for continuous random variables.

**Definition**

The cumulative distribution function of a continuous random variable $X$ is

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(u) \, du$$  \hspace{1cm} (4-3)$$

for $-\infty < x < \infty$.

Extending the definition of $f(x)$ to the entire real line enables us to define the cumulative distribution function for all real numbers. The following example illustrates the definition.

**EXAMPLE 4-3**

For the copper current measurement in Example 4-1, the cumulative distribution function of the random variable $X$ consists of three expressions. If $x < 0$, $f(x) = 0$. Therefore,

$$F(x) = 0, \quad \text{for} \quad x < 0$$
and

\[ F(x) = \int_0^x f(u) \, du = 0.05x, \quad \text{for} \quad 0 \leq x < 20 \]

Finally,

\[ F(x) = \int_0^x f(u) \, du = 1, \quad \text{for} \quad 20 \leq x \]

Therefore,

\[ F(x) = \begin{cases} 
0 & x < 0 \\
0.05x & 0 \leq x < 20 \\
1 & 20 \leq x 
\end{cases} \]

The plot of \( F(x) \) is shown in Fig. 4-6.

Notice that in the definition of \( F(x) \) any \( < \) can be changed to \( \leq \) and vice versa. That is, \( F(x) \) can be defined as either \( 0.05x \) or \( 0 \) at the end-point \( x = 0 \), and \( F(x) \) can be defined as either \( 0.05x \) or \( 1 \) at the end-point \( x = 20 \). In other words, \( F(x) \) is a continuous function. For a discrete random variable, \( F(x) \) is not a continuous function. Sometimes, a continuous random variable is defined as one that has a continuous cumulative distribution function.

**EXAMPLE 4-4**

For the drilling operation in Example 4-2, \( F(x) \) consists of two expressions.

\[ F(x) = 0 \quad \text{for} \quad x < 12.5 \]

and for \( 12.5 \leq x \)

\[ F(x) = \int_0^x 20e^{-20(u-12.5)} \, du = 1 - e^{-20(x-12.5)} \]

Therefore,

\[ F(x) = \begin{cases} 
0 & x < 12.5 \\
1 - e^{-20(x-12.5)} & 12.5 \leq x 
\end{cases} \]

Figure 4-7 displays a graph of \( F(x) \).
The probability density function of a continuous random variable can be determined from the cumulative distribution function by differentiating. Recall that the fundamental theorem of calculus states that
\[ \frac{d}{dx} \int_{-\infty}^{x} f(u) \, du = f(x) \]

Then, given \( F(x) \)
\[ f(x) = \frac{dF(x)}{dx} \]
as long as the derivative exists.

**EXAMPLE 4-5**  
The time until a chemical reaction is complete (in milliseconds) is approximated by the cumulative distribution function
\[ F(x) = \begin{cases} 
0 & \text{if } x < 0 \\
1 - e^{-0.01x} & \text{if } 0 \leq x 
\end{cases} \]

Determine the probability density function of \( X \). What proportion of reactions is complete within 200 milliseconds? Using the result that the probability density function is the derivative of the \( F(x) \), we obtain
\[ f(x) = \begin{cases} 
0 & \text{if } x < 0 \\
0.01e^{-0.01x} & \text{if } 0 \leq x 
\end{cases} \]
The probability that a reaction completes within 200 milliseconds is
\[ P(X < 200) = F(200) = 1 - e^{-2} = 0.8647. \]

**EXERCISES FOR SECTION 4-3**

4-11. Suppose the cumulative distribution function of the random variable \( X \) is
\[ F(x) = \begin{cases} 
0 & \text{if } x < 0 \\
0.2x & \text{if } 0 \leq x < 5 \\
1 & \text{if } 5 \leq x 
\end{cases} \]

Determine the following:
(a) \( P(X < 2.8) \)  
(b) \( P(X > 1.5) \)  
(c) \( P(X < -2) \)  
(d) \( P(X > 6) \)

4-12. Suppose the cumulative distribution function of the random variable \( X \) is
\[ F(x) = \begin{cases} 
0 & \text{if } x < -2 \\
0.25x + 0.5 & \text{if } -2 \leq x < 2 \\
1 & \text{if } 2 \leq x 
\end{cases} \]

Determine the following:
(a) \( P(X < 1.8) \)  
(b) \( P(X > -1.5) \)  
(c) \( P(X < -2) \)  
(d) \( P(-1 < X < 1) \)

4-13. Determine the cumulative distribution function for the distribution in Exercise 4-1.

4-14. Determine the cumulative distribution function for the distribution in Exercise 4-3.

4-15. Determine the cumulative distribution function for the distribution in Exercise 4-4.

4-16. Determine the cumulative distribution function for the distribution in Exercise 4-6. Use the cumulative distribution function to determine the probability that a component lasts more than 3000 hours before failure.

4-17. Determine the cumulative distribution function for the distribution in Exercise 4-8. Use the cumulative distribution function to determine the probability that a length exceeds 75 millimeters.
4-4 MEAN AND VARIANCE OF A CONTINUOUS RANDOM VARIABLE

The mean and variance of a continuous random variable are defined similarly to a discrete random variable. Integration replaces summation in the definitions. If a probability density function is viewed as a loading on a beam as in Fig. 4-1, the mean is the balance point.

Determine the probability density function for each of the following cumulative distribution functions.

4-18. \( F(x) = 1 - e^{-2x} \quad x > 0 \)

4-19. \( F(x) = \begin{cases} 
0 & x < 0 \\
0.2x & 0 \leq x < 4 \\
0.04x + 0.64 & 4 \leq x < 9 \\
1 & 9 \leq x 
\end{cases} \)

4-20. \( F(x) = \begin{cases} 
0 & x < -2 \\
0.25x + 0.5 & -2 \leq x < 1 \\
0.5x + 0.25 & 1 \leq x < 1.5 \\
1 & 1.5 \leq x 
\end{cases} \)

4-21. The gap width is an important property of a magnetic recording head. In coded units, if the width is a continuous random variable over the range from 0 < x < 2 with \( f(x) = 0.5x \), determine the cumulative distribution function of the gap width.

4-4 MEAN AND VARIANCE OF A CONTINUOUS RANDOM VARIABLE

Suppose \( X \) is a continuous random variable with probability density function \( f(x) \). The mean or expected value of \( X \), denoted as \( \mu \) or \( E(X) \), is

\[
\mu = E(X) = \int_{-\infty}^{\infty} xf(x) \, dx \quad (4-4)
\]

The variance of \( X \), denoted as \( V(X) \) or \( \sigma^2 \), is

\[
\sigma^2 = V(X) = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx = \int_{-\infty}^{\infty} x^2 f(x) \, dx - \mu^2
\]

The standard deviation of \( X \) is \( \sigma = \sqrt{\sigma^2} \).

The equivalence of the two formulas for variance can be derived as one, as was done for discrete random variables.

EXAMPLE 4-6

For the copper current measurement in Example 4-1, the mean of \( X \) is

\[
E(X) = \int_{0}^{20} xf(x) \, dx = 0.05x^2/2 \bigg|_{0}^{20} = 10
\]

The variance of \( X \) is

\[
v(X) = \int_{0}^{20} (x - 10)^2 f(x) \, dx = 0.05(x - 10)^3/3 \bigg|_{0}^{20} = 33.33
\]
The expected value of a function \( h(X) \) of a continuous random variable is defined similarly to a function of a discrete random variable.

\[
E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x) \, dx \tag{4-5}
\]

**EXAMPLE 4-7**

In Example 4-1, \( X \) is the current measured in milliamperes. What is the expected value of the squared current? Now, \( E[X^2] = \int_{-\infty}^{\infty} x^2 f(x) \, dx \). Therefore,

\[
E[X^2] = \int_{0}^{20} 0.05x^2 \, dx = 0.05 \int_{0}^{20} \frac{x^3}{3} \, dx = 133.33
\]

In the previous example, the expected value of \( X^2 \) does not equal \( E(X)^2 \). However, in the special case that \( h(X) = aX + b \) for any constants \( a \) and \( b \), \( E[h(X)] = aE(X) + b \). This can be shown from the properties of integrals.

**EXAMPLE 4-8**

For the drilling operation in Example 4-2, the mean of \( X \) is

\[
E(X) = \int_{12.5}^{\infty} x f(x) \, dx = \int_{12.5}^{\infty} 20e^{-20(x-12.5)} \, dx
\]

Integration by parts can be used to show that

\[
E(X) = -xe^{-20(x-12.5)} - \frac{e^{-20(x-12.5)}}{20} \bigg|_{12.5}^{\infty} = 12.5 + 0.05 = 12.55
\]

The variance of \( X \) is

\[
V(X) = \int_{12.5}^{\infty} (x - 12.55)^2 f(x) \, dx
\]

Although more difficult, integration by parts can be used two times to show that \( V(X) = 0.0025 \).

**EXERCISES FOR SECTION 4.4**

4-22. Suppose \( f(x) = 0.25 \) for \( 0 < x < 4 \). Determine the mean and variance of \( X \).

4-23. Suppose \( f(x) = 0.125x \) for \( 0 < x < 4 \). Determine the mean and variance of \( X \).

4-24. Suppose \( f(x) = 1.5x^2 \) for \( -1 < x < 1 \). Determine the mean and variance of \( X \).

4-25. Suppose that \( f(x) = x/8 \) for \( 3 < x < 5 \). Determine the mean and variance for \( x \).

4-26. Determine the mean and variance of the weight of packages in Exercise 4.7.

4-27. The thickness of a conductive coating in micrometers has a density function of \( 600x^{-2} \) for \( 100 \mu m < x < 120 \mu m \).
(a) Determine the mean and variance of the coating thickness.
(b) If the coating costs $0.50 per micrometer of thickness on each part, what is the average cost of the coating per part?

4-28. Suppose that contamination particle size (in micrometers) can be modeled as \( f(x) = 2x^{-3} \) for \( 1 < x \). Determine the mean of \( X \).

4-29. Integration by parts is required. The probability density function for the diameter of a drilled hole in millimeters is \( 10e^{-10(x-5)} \) for \( x > 5 \) mm. Although the target diameter is 5 millimeters, vibrations, tool wear, and other nuisances produce diameters larger than 5 millimeters.

### 4-5 CONTINUOUS UNIFORM DISTRIBUTION

The simplest continuous distribution is analogous to its discrete counterpart.

**Definition**

A continuous random variable \( X \) with probability density function

\[
f(x) = \frac{1}{b - a}, \quad a \leq x \leq b
\]

is a **continuous uniform random variable**.

The probability density function of a continuous uniform random variable is shown in Fig. 4-8. The mean of the continuous uniform random variable \( X \) is

\[
E(X) = \int_a^b \frac{x}{b - a} \, dx = \left. \frac{0.5x^2}{b - a} \right|_a^b = \frac{(a + b)}{2}
\]

The variance of \( X \) is

\[
V(X) = \int_a^b \left( \frac{x - \frac{a + b}{2}}{b - a} \right)^2 \, dx = \left. \frac{(x - \frac{a + b}{2})^3}{3(b - a)} \right|_a^b = \frac{(b - a)^2}{12}
\]

These results are summarized as follows.

If \( X \) is a continuous uniform random variable over \( a \leq x \leq b \),

\[
\mu = E(X) = \frac{(a + b)}{2} \quad \text{and} \quad \sigma^2 = V(X) = \frac{(b - a)^2}{12}
\]

(4-7)
EXAMPLE 4-9

Let the continuous random variable $X$ denote the current measured in a thin copper wire in milliamperes. Assume that the range of $X$ is $[0, 20\ mA]$, and assume that the probability density function of $X$ is

What is the probability that a measurement of current is between 5 and 10 milliamperes?

The requested probability is shown as the shaded area in Fig. 4-9.

$$P(5 < X < 10) = \int_{5}^{10} f(x) \, dx = 5(0.05) = 0.25$$

The mean and variance formulas can be applied with $a = 0$ and $b = 20$. Therefore,

$$E(X) = 10\ mA \quad \text{and} \quad V(X) = 20^2/12 = 33.33\ mA^2$$

Consequently, the standard deviation of $X$ is 5.77 mA.

The cumulative distribution function of a continuous uniform random variable is obtained by integration. If $a < x < b$,

$$F(x) = \frac{x}{b - a}$$

Therefore, the complete description of the cumulative distribution function of a continuous uniform random variable is

$$F(x) = \begin{cases} 
0 & x < a \\
(x - a)/(b - a) & a \leq x < b \\
1 & b \leq x
\end{cases}$$

An example of $F(x)$ for a continuous uniform random variable is shown in Fig. 4-6.

EXERCISES FOR SECTION 4-5

4-31. Suppose $X$ has a continuous uniform distribution over the interval $[1.5, 5.5]$.

(a) Determine the mean, variance, and standard deviation of $X$.

(b) What is $P(X < 2.5)$?

4-32. Suppose $X$ has a continuous uniform distribution over the interval $[-1, 1]$.

(a) Determine the mean, variance, and standard deviation of $X$.

(b) Determine the value for $x$ such that $P(-x < X < x) = 0.90$.

4-33. The net weight in pounds of a packaged chemical herbicide is uniform for $49.75 < x < 50.25$ pounds.

(a) Determine the mean and variance of the weight of packages.
Undoubtedly, the most widely used model for the distribution of a random variable is a normal distribution. Whenever a random experiment is replicated, the random variable that equals the average (or total) result over the replicates tends to have a normal distribution as the number of replicates becomes large. De Moivre presented this fundamental result, known as the central limit theorem, in 1733. Unfortunately, his work was lost for some time, and Gauss independently developed a normal distribution nearly 100 years later. Although De Moivre was later credited with the derivation, a normal distribution is also referred to as a Gaussian distribution.

When do we average (or total) results? Almost always. For example, an automotive engineer may plan a study to average pull-off force measurements from several connectors. If we assume that each measurement results from a replicate of a random experiment, the normal distribution can be used to make approximate conclusions about this average. These conclusions are the primary topics in the subsequent chapters of this book.

Furthermore, sometimes the central limit theorem is less obvious. For example, assume that the deviation (or error) in the length of a machined part is the sum of a large number of infinitesimal effects, such as temperature and humidity drifts, vibrations, cutting angle variations, cutting tool wear, bearing wear, rotational speed variations, mounting and fixturing variations, variations in numerous raw material characteristics, and variation in levels of contamination. If the component errors are independent and equally likely to be positive or negative, the total error can be shown to have an approximate normal distribution. Furthermore, the normal distribution arises in the study of numerous basic physical phenomena. For example, the physicist Maxwell developed a normal distribution from simple assumptions regarding the velocities of molecules.

The theoretical basis of a normal distribution is mentioned to justify the somewhat complex form of the probability density function. Our objective now is to calculate probabilities for a normal random variable. The central limit theorem will be stated more carefully later.
Random variables with different means and variances can be modeled by normal probability density functions with appropriate choices of the center and width of the curve. The value of $E(X) = \mu$ determines the center of the probability density function and the value of $V(X) = \sigma^2$ determines the width. Figure 4-10 illustrates several normal probability density functions with selected values of $\mu$ and $\sigma^2$. Each has the characteristic symmetric bell-shaped curve, but the centers and dispersions differ. The following definition provides the formula for normal probability density functions.

**Definition**

A random variable $X$ with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi \sigma}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \quad -\infty < x < \infty \quad (4-8)$$

is a normal random variable with parameters $\mu$, where $-\infty < \mu < \infty$, and $\sigma > 0$. Also,

$$E(X) = \mu \quad \text{and} \quad V(X) = \sigma^2 \quad (4-9)$$

and the notation $N(\mu, \sigma^2)$ is used to denote the distribution. The mean and variance of $X$ are shown to equal $\mu$ and $\sigma^2$, respectively, at the end of this Section 5-6.

**EXAMPLE 4-10**

Assume that the current measurements in a strip of wire follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)$^2$. What is the probability that a measurement exceeds 13 milliamperes?

Let $X$ denote the current in milliamperes. The requested probability can be represented as $P(X > 13)$. This probability is shown as the shaded area under the normal probability density function in Fig. 4-11. Unfortunately, there is no closed-form expression for the integral of a normal probability density function, and probabilities based on the normal distribution are typically found numerically or from a table (that we will later introduce).

Some useful results concerning a normal distribution are summarized below and in Fig. 4-12. For any normal random variable,

$$P(\mu - \sigma < X < \mu + \sigma) = 0.6827$$
$$P(\mu - 2\sigma < X < \mu + 2\sigma) = 0.9545$$
$$P(\mu - 3\sigma < X < \mu + 3\sigma) = 0.9973$$

Also, from the symmetry of $f(x)$, $P(X > \mu) = P(X < \mu) = 0.5$. Because $f(x)$ is positive for all $x$, this model assigns some probability to each interval of the real line. However, the
probability density function decreases as \(x\) moves farther from \(\mu\). Consequently, the probability that a measurement falls far from \(\mu\) is small, and at some distance from \(\mu\) the probability of an interval can be approximated as zero.

The area under a normal probability density function beyond 3\(\sigma\) from the mean is quite small. This fact is convenient for quick, rough sketches of a normal probability density function. The sketches help us determine probabilities. Because more than 0.9973 of the probability of a normal distribution is within the interval \((\mu - 3\sigma, \mu + 3\sigma)\), 6\(\sigma\) is often referred to as the width of a normal distribution. Advanced integration methods can be used to show that the area under the normal probability density function from \(-\infty\) to \(\infty\) is 1.

**Definition**

A normal random variable with

\[
\mu = 0 \quad \text{and} \quad \sigma^2 = 1
\]

is called a **standard normal random variable** and is denoted as \(Z\).

The cumulative distribution function of a standard normal random variable is denoted as

\[
\Phi(z) = P(Z \leq z)
\]

Appendix Table II provides cumulative probability values for \(\Phi(z)\), for a standard normal random variable. Cumulative distribution functions for normal random variables are also widely available in computer packages. They can be used in the same manner as Appendix Table II to obtain probabilities for these random variables. The use of Table II is illustrated by the following example.

**EXAMPLE 4-11**

Assume \(Z\) is a standard normal random variable. Appendix Table II provides probabilities of the form \(P(Z \leq z)\). The use of Table II to find \(P(Z \leq 1.5)\) is illustrated in Fig. 4-13. Read down the \(z\) column to the row that equals 1.5. The probability is read from the adjacent column, labeled 0.00, to be 0.93319.

The column headings refer to the hundredth's digit of the value of \(z\) in \(P(Z \leq z)\). For example, \(P(Z \leq 1.53)\) is found by reading down the \(z\) column to the row 1.5 and then selecting the probability from the column labeled 0.03 to be 0.93699.
Probabilities that are not of the form \( P(Z \leq z) \) are found by using the basic rules of probability and the symmetry of the normal distribution along with Appendix Table II. The following examples illustrate the method.

**EXAMPLE 4-12**

The following calculations are shown pictorially in Fig. 4-14. In practice, a probability is often rounded to one or two significant digits.

1. \( P(Z > 1.26) = 1 - P(Z \leq 1.26) = 1 - 0.89616 = 0.10384 \)
2. \( P(Z < -0.86) = 0.19490 \).
3. \( P(Z > -1.37) = P(Z < 1.37) = 0.91465 \)
4. \( P(-1.25 < Z < 0.37) \). This probability can be found from the difference of two areas, \( P(Z < 0.37) - P(Z < -1.25) \). Now,

\[
P(Z < 0.37) = 0.64431 \quad \text{and} \quad P(Z < -1.25) = 0.10565
\]

Therefore,

\[
P(-1.25 < Z < 0.37) = 0.64431 - 0.10565 = 0.53866
\]
(5) \( P(Z \leq -4.6) \) cannot be found exactly from Appendix Table II. However, the last entry in the table can be used to find that \( P(Z \leq -3.99) = 0.00003 \). Because \( P(Z \leq -4.6) < P(Z \leq -3.99) \), \( P(Z \leq -4.6) \) is nearly zero.

(6) Find the value \( z \) such that \( P(Z > z) = 0.05 \). This probability expression can be written as \( P(Z \leq z) = 0.95 \). Now, Table II is used in reverse. We search through the probabilities to find the value that corresponds to 0.95. The solution is illustrated in Fig. 4-14. We do not find 0.95 exactly; the nearest value is 0.95053, corresponding to \( z = 1.65 \).

(7) Find the value of \( z \) such that \( P(-z < Z < z) = 0.99 \). Because of the symmetry of the normal distribution, if the area of the shaded region in Fig. 4-14(7) is to equal 0.99, the area in each tail of the distribution must equal 0.005. Therefore, the value for \( z \) corresponds to a probability of 0.995 in Table II. The nearest probability in Table II is 0.99506, when \( z = 2.58 \).

The preceding examples show how to calculate probabilities for standard normal random variables. To use the same approach for an arbitrary normal random variable would require a separate table for every possible pair of values for \( \mu \) and \( \sigma \). Fortunately, all normal probability distributions are related algebraically, and Appendix Table II can be used to find the probabilities associated with an arbitrary normal random variable by first using a simple transformation.

If \( X \) is a normal random variable with \( E(X) = \mu \) and \( V(X) = \sigma^2 \), the random variable

\[
Z = \frac{X - \mu}{\sigma}
\]  

is a normal random variable with \( E(Z) = 0 \) and \( V(Z) = 1 \). That is, \( Z \) is a standard normal random variable.

Creating a new random variable by this transformation is referred to as **standardizing**. The random variable \( Z \) represents the distance of \( X \) from its mean in terms of standard deviations. It is the key step to calculate a probability for an arbitrary normal random variable.

**EXAMPLE 4-13**

Suppose the current measurements in a strip of wire are assumed to follow a normal distribution with a mean of 10 milliamperes and a variance of 4 (milliamperes)^2. What is the probability that a measurement will exceed 13 milliamperes?

Let \( X \) denote the current in milliamperes. The requested probability can be represented as \( P(X > 13) \). Let \( Z = (X - 10)/2 \). The relationship between the several values of \( X \) and the transformed values of \( Z \) are shown in Fig. 4-15. We note that \( X > 13 \) corresponds to \( Z > 1.5 \). Therefore, from Appendix Table II,

\[
P(X > 13) = P(Z > 1.5) = 1 - P(Z \leq 1.5) = 1 - 0.93319 = 0.06681
\]

Rather than using Fig. 4-15, the probability can be found from the inequality \( X > 13 \). That is,

\[
P(X > 13) = P\left( \frac{(X - 10)}{2} > \frac{(13 - 10)}{2} \right) = P(Z > 1.5) = 0.06681
\]
Suppose $X$ is a normal random variable with mean $\mu$ and variance $\sigma^2$. Then,

$$P(X \leq x) = P\left(\frac{X - \mu}{\sigma} \leq \frac{x - \mu}{\sigma}\right) = P(Z \leq z) \tag{4-11}$$

where $Z$ is a standard normal random variable, and $z = \frac{x - \mu}{\sigma}$ is the $z$-value obtained by standardizing $X$.

The probability is obtained by entering Appendix Table II with $z = \frac{x - \mu}{\sigma}$.

**EXAMPLE 4-14** Continuing the previous example, what is the probability that a current measurement is between 9 and 11 milliamperes? From Fig. 4-15, or by proceeding algebraically, we have

$$P(9 < X < 11) = P(9 - 10)/2 < (X - 10)/2 < (11 - 10)/2$$

$$= P(-0.5 < Z < 0.5) = P(Z < 0.5) - P(Z < -0.5)$$

$$= 0.69146 - 0.30854 = 0.38292$$

Determine the value for which the probability that a current measurement is below this value is 0.98. The requested value is shown graphically in Fig. 4-16. We need the value of $x$ such that $P(X < x) = 0.98$. By standardizing, this probability expression can be written as

$$P(X < x) = P(X - 10)/2 < (x - 10)/2$$

$$= P(Z < (x - 10)/2)$$

$$= 0.98$$

Appendix Table II is used to find the $z$-value such that $P(Z < z) = 0.98$. The nearest probability from Table II results in

$$P(Z < 2.05) = 0.97982$$
Therefore, \((x - 10)/2 = 2.05\), and the standardizing transformation is used in reverse to solve for \(x\). The result is

\[ x = 2(2.05) + 10 = 14.1 \text{ milliamperes} \]

**EXAMPLE 4-15**  
Assume that in the detection of a digital signal the background noise follows a normal distribution with a mean of 0 volt and standard deviation of 0.45 volt. The system assumes a digital 1 has been transmitted when the voltage exceeds 0.9. What is the probability of detecting a digital 1 when none was sent?

Let the random variable \(N\) denote the voltage of noise. The requested probability is

\[ P(N > 0.9) = P\left( \frac{N}{0.45} > \frac{0.9}{0.45} \right) = P(Z > 2) = 1 - 0.97725 = 0.02275 \]

This probability can be described as the probability of a false detection.

Determine symmetric bounds about 0 that include 99% of all noise readings. The question requires us to find \(x\) such that \(P(-x < N < x) = 0.99\). A graph is shown in Fig. 4-17. Now,

\[ P(-x < N < x) = P\left( -x/0.45 < N/0.45 < x/0.45 \right) = P\left( -x/0.45 < Z < x/0.45 \right) = 0.99 \]

From Appendix Table II

\[ P(-2.58 < Z < 2.58) = 0.99 \]
Therefore,

\[ x/0.45 = 2.58 \]

and

\[ x = 2.58(0.45) = 1.16 \]

Suppose a digital 1 is represented as a shift in the mean of the noise distribution to 1.8 volts. What is the probability that a digital 1 is not detected? Let the random variable \( S \) denote the voltage when a digital 1 is transmitted. Then,

\[ P(S < 0.9) = P\left( \frac{S - 1.8}{0.45} < \frac{0.9 - 1.8}{0.45} \right) = P(Z < -2) = 0.02275 \]

This probability can be interpreted as the probability of a missed signal.

**EXAMPLE 4-16** 

The diameter of a shaft in an optical storage drive is normally distributed with mean 0.2508 inch and standard deviation 0.0005 inch. The specifications on the shaft are 0.2500 ± 0.0015 inch. What proportion of shafts conforms to specifications?

Let \( X \) denote the shaft diameter in inches. The requested probability is shown in Fig. 4-18 and

\[
P(0.2485 < X < 0.2515) = P\left( \frac{0.2485 - 0.2508}{0.0005} < Z < \frac{0.2515 - 0.2508}{0.0005} \right)
\]

\[= P(-4.6 < Z < 1.4) = P(Z < 1.4) - P(Z < -4.6)
\]

\[= 0.91924 - 0.0000 = 0.91924 \]

Most of the nonconforming shafts are too large, because the process mean is located very near to the upper specification limit. If the process is centered so that the process mean is equal to the target value of 0.2500,

\[
P(0.2485 < X < 0.2515) = P\left( \frac{0.2485 - 0.2500}{0.0005} < Z < \frac{0.2515 - 0.2500}{0.0005} \right)
\]

\[= P(-3 < Z < 3)
\]

\[= P(Z < 3) - P(Z < -3)
\]

\[= 0.99865 - 0.00135
\]

\[= 0.9973 \]

By recentering the process, the yield is increased to approximately 99.73%.

![Figure 4-18](image_url) 

Distribution for Example 4-16.
EXERCISES FOR SECTION 4-6

4-39. Use Appendix Table II to determine the following probabilities for the standard normal random variable Z:
(a) $P(Z < 1.32)$
(b) $P(Z < 3.0)$
(c) $P(Z > 1.45)$
(d) $P(Z > -2.15)$
(e) $P(-2.34 < Z < 1.76)$

4-40. Use Appendix Table II to determine the following probabilities for the standard normal random variable Z:
(a) $P(-1 < Z < 1)$
(b) $P(-2 < Z < 2)$
(c) $P(-3 < Z < 3)$
(d) $P(Z > 3)$
(e) $P(0 < Z < 1)$

4-41. Assume $Z$ has a standard normal distribution. Use Appendix Table II to determine the value for $z$ that solves each of the following:
(a) $P(Z < z) = 0.9$
(b) $P(Z < z) = 0.5$
(c) $P(Z > z) = 0.1$
(d) $P(Z > z) = 0.9$
(e) $P(1.24 < Z < z) = 0.8$

4-42. Assume $Z$ has a standard normal distribution. Use Appendix Table II to determine the value for $z$ that solves each of the following:
(a) $P(-z < Z < z) = 0.95$
(b) $P(-z < Z < z) = 0.99$
(c) $P(-z < Z < z) = 0.68$
(d) $P(-z < Z < z) = 0.9773$

4-43. Assume $X$ is normally distributed with a mean of 10 and a standard deviation of 2. Determine the following:
(a) $P(X < 13)$
(b) $P(X > 9)$
(c) $P(6 < X < 14)$
(d) $P(2 < X < 4)$
(e) $P(2 < X < 8)$

4-44. Assume $X$ is normally distributed with a mean of 10 and a standard deviation of 2. Determine the value for $x$ that solves each of the following:
(a) $P(X > x) = 0.5$
(b) $P(X > x) = 0.95$
(c) $P(x < X < 10) = 0.2$
(d) $P(-x < X < 10 - x) = 0.95$
(e) $P(-x < X < 10 - x) = 0.99$

4-45. Assume $X$ is normally distributed with a mean of 5 and a standard deviation of 4. Determine the following:
(a) $P(X < 11)$
(b) $P(X > 0)$
(c) $P(3 < X < 7)$
(d) $P(-2 < X < 9)$
(e) $P(2 < X < 8)$

4-46. Assume $X$ is normally distributed with a mean of 5 and a standard deviation of 4. Determine the value for $x$ that solves each of the following:
(a) $P(X > x) = 0.5$
(b) $P(X > x) = 0.95$
(c) $P(x < X < 9) = 0.2$
(d) $P(3 < X < x) = 0.95$
(e) $P(-x < X < x) = 0.99$

4-47. The compressive strength of samples of cement can be modeled by a normal distribution with a mean of 6000 kilograms per square centimeter and a standard deviation of 100 kilograms per square centimeter.

(a) What is the probability that a sample’s strength is less than 6250 Kg/cm²?
(b) What is the probability that a sample’s strength is between 5800 and 5900 Kg/cm²?
(c) What strength is exceeded by 95% of the samples?

4-48. The compressive strength of cement can be modeled by a normal distribution with a mean of 35 pounds per square inch and a standard deviation of 2 pounds per square inch.
(a) What is the probability that the strength of a sample is less than 40 lb/in²?
(b) If the specifications require the tensile strength to exceed 30 lb/in², what proportion of the samples is scrapped?

4-49. The line width of semiconductor manufacturing is assumed to be normally distributed with a mean of 0.5 micrometer and a standard deviation of 0.05 micrometer.
(a) What is the probability that a line width is greater than 0.62 micrometer?
(b) What is the probability that a line width is between 0.47 and 0.63 micrometer?
(c) The line width of 90% of samples is below what value?

4-50. The fill volume of an automated filling machine used for filling cans of carbonated beverage is normally distributed with a mean of 12.4 fluid ounces and a standard deviation of 0.1 fluid ounce.
(a) What is the probability a fill volume is less than 12 fluid ounces?
(b) If all cans less than 12.1 or greater than 12.6 ounces are scrapped, what proportion of cans is scrapped?
(c) Determine specifications that are symmetric about the mean that include 99% of all cans.

4-51. The time it takes a cell to divide (called mitosis) is normally distributed with an average time of one hour and a standard deviation of 5 minutes.
(a) What is the probability that a cell divides in less than 45 minutes?
(b) What is the probability that it takes a cell more than 65 minutes to divide?
(c) What is the time that it takes approximately 99% of all cells to complete mitosis?

4-52. In the previous exercise, suppose that the mean of the filling operation can be adjusted easily, but the standard deviation remains at 0.1 ounce.
(a) At what value should the mean be set so that 99.9% of all cans exceed 12 ounces?
(b) At what value should the mean be set so that 99.9% of all cans exceed 12 ounces if the standard deviation can be reduced to 0.05 fluid ounce?
The reaction time of a driver to visual stimulus is normally distributed with a mean of 0.4 seconds and a standard deviation of 0.05 seconds.

(a) What is the probability that a reaction requires more than 0.5 seconds?
(b) What is the probability that a reaction requires between 0.4 and 0.5 seconds?
(c) What is the reaction time that is exceeded 90% of the time?

The speed of a file transfer from a server on campus to a personal computer at a student’s home on a weekday evening is normally distributed with a mean of 60 kilobits per second and a standard deviation of 4 kilobits per second.

(a) What is the probability that the file will transfer at a speed of 70 kilobits per second or more?
(b) What is the probability that the file will transfer at a speed of less than 58 kilobits per second?
(c) If the file is 1 megabyte, what is the average time it will take to transfer the file? (Assume eight bits per byte.)

The length of an injection-molded plastic case that holds magnetic tape is normally distributed with a length of 90.2 millimeters and a standard deviation of 0.1 millimeter.

(a) What is the probability that a part is longer than 90.3 millimeters or shorter than 89.7 millimeters?
(b) What should the process mean be set at to obtain the greatest number of parts between 89.7 and 90.3 millimeters?
(c) If parts that are not between 89.7 and 90.3 millimeters are scrapped, what is the yield for the process mean that you selected in part (b)?

In the previous exercise assume that the process is centered so that the mean is 90 millimeters and the standard deviation is 0.1 millimeter. Suppose that 10 cases are measured, and they are assumed to be independent.

(a) What is the probability that all 10 cases are between 89.7 and 90.3 millimeters?
(b) What is the expected number of the 10 cases that are between 89.7 and 90.3 millimeters?

The sick-leave time of employees in a firm in a month is normally distributed with a mean of 100 hours and a standard deviation of 20 hours.

(a) What is the probability that the sick-leave time for next month will be between 50 and 80 hours?
(b) How much time should be budgeted for sick leave if the budgeted amount should be exceeded with a probability of only 10%?

The life of a semiconductor laser at a constant power is normally distributed with a mean of 7000 hours and a standard deviation of 600 hours.

(a) What is the probability that a laser fails before 5000 hours?
(b) What is the life in hours that 95% of the lasers exceed?
(c) If three lasers are used in a product and they are assumed to fail independently, what is the probability that all three are still operating after 7000 hours?

The diameter of the dot produced by a printer is normally distributed with a mean diameter of 0.002 inch and a standard deviation of 0.0004 inch.

(a) What is the probability that the diameter of a dot exceeds 0.0026 inch?
(b) What is the probability that a diameter is between 0.0014 and 0.0026 inch?
(c) What standard deviation of diameters is needed so that the probability in part (b) is 0.995?

The weight of a sophisticated running shoe is normally distributed with a mean of 12 ounces and a standard deviation of 0.5 ounce.

(a) What is the probability that a shoe weighs more than 13 ounces?
(b) What must the standard deviation of weight be in order for the company to state that 99.9% of its shoes are less than 13 ounces?
(c) If the standard deviation remains at 0.5 ounce, what must the mean weight be in order for the company to state that 99.9% of its shoes are less than 13 ounces?

The reaction time of a driver to visual stimulus is normally distributed with a mean of 0.4 seconds and a standard deviation of 0.05 seconds.

(a) What is the probability that a reaction requires more than 0.5 seconds?
(b) What is the probability that a reaction requires between 0.4 and 0.5 seconds?
(c) What is the reaction time that is exceeded 90% of the time?

4-55. The speed of a file transfer from a server on campus to a personal computer at a student’s home on a weekday evening is normally distributed with a mean of 60 kilobits per second and a standard deviation of 4 kilobits per second.

(a) What is the probability that the file will transfer at a speed of 70 kilobits per second or more?
(b) What is the probability that the file will transfer at a speed of less than 58 kilobits per second?
(c) If the file is 1 megabyte, what is the average time it will take to transfer the file? (Assume eight bits per byte.)

4-56. In the previous exercise assume that the process is centered so that the mean is 90 millimeters and the standard deviation is 0.1 millimeter. Suppose that 10 cases are measured, and they are assumed to be independent.

(a) What is the probability that all 10 cases are between 89.7 and 90.3 millimeters?
(b) What is the expected number of the 10 cases that are between 89.7 and 90.3 millimeters?

4-57. The sick-leave time of employees in a firm in a month is normally distributed with a mean of 100 hours and a standard deviation of 20 hours.

(a) What is the probability that the sick-leave time for next month will be between 50 and 80 hours?
(b) How much time should be budgeted for sick leave if the budgeted amount should be exceeded with a probability of only 10%?

4-58. The life of a semiconductor laser at a constant power is normally distributed with a mean of 7000 hours and a standard deviation of 600 hours.

(a) What is the probability that a laser fails before 5000 hours?
(b) What is the life in hours that 95% of the lasers exceed?
(c) If three lasers are used in a product and they are assumed to fail independently, what is the probability that all three are still operating after 7000 hours?

4-59. The diameter of the dot produced by a printer is normally distributed with a mean diameter of 0.002 inch and a standard deviation of 0.0004 inch.

(a) What is the probability that the diameter of a dot exceeds 0.0026 inch?
(b) What is the probability that a diameter is between 0.0014 and 0.0026 inch?
(c) What standard deviation of diameters is needed so that the probability in part (b) is 0.995?

4-60. The weight of a sophisticated running shoe is normally distributed with a mean of 12 ounces and a standard deviation of 0.5 ounce.

(a) What is the probability that a shoe weighs more than 13 ounces?
(b) What must the standard deviation of weight be in order for the company to state that 99.9% of its shoes are less than 13 ounces?
(c) If the standard deviation remains at 0.5 ounce, what must the mean weight be in order for the company to state that 99.9% of its shoes are less than 13 ounces?

4-7 NORMAL APPROXIMATION TO THE BINOMIAL AND POISSON DISTRIBUTIONS

We began our section on the normal distribution with the central limit theorem and the normal distribution as an approximation to a random variable with a large number of trials. Consequently, it should not be a surprise to learn that the normal distribution can be used to approximate binomial probabilities for cases in which n is large. The following example illustrates that for many physical systems the binomial model is appropriate with an extremely large value for n. In these cases, it is difficult to calculate probabilities by using the binomial distribution. Fortunately, the normal approximation is most effective in these cases. An illustration is provided in Fig. 4-19. The area of each bar equals the binomial probability of x. Notice that the area of bars can be approximated by areas under the normal density function.