Section 1.1 Differential Equations and Mathematical Models, cont’d.

Obtaining differential equations from geometric conditions

• In some problems, we are given a geometric property of the graph of an unknown function \( g(x) \).
• We are then asked to determine what \( g(x) \) is.
• The first step in finding \( g(x) \) is to translate its geometric properties into a differential equation.
• For now we will focus only on this first step.

**Example 1:** The line tangent to the graph of \( g(x) \) at an arbitrary point \((x, y)\) passes through the point \((1, 2)\). Write a differential equation of the form

\[ \frac{dy}{dx} = f(x, y) \]

having the function \( g \) as its solution (or one of its solutions).
Example 2: Every line normal to the graph of \( g(x) \) passes through the point \((1,0)\). Again, write a differential equation of the form

\[
\frac{dy}{dx} = f(x,y)
\]

having the function \( g \) as its solution (or one of its solutions).
**Obtaining differential equations from situations**

- Now we are going to attempt to do what is done in the "real world": Take the verbal description of a situation, and rewrite the description using a differential equation.
- Usually this step is harder than actually solving the differential equation.

**Example 3:** An English clergyman named Thomas Malthus argued that the rate of increase of the world’s population at any time was proportional to its size at that time. This rule is called *Malthus’ growth principle*. Restate Malthus’ growth principle using a differential equation.

**Example 4:** In a city having a fixed population of $P$ persons, the time rate of change of the number $N$ of people who have heard a certain rumor is proportional to the number of those who have not yet heard the rumor. Write a differential equation that is a mathematical model of this situation.

**Solving differential equations using inspection**

"Inspection" just means "guessing".

- Sometimes the derivatives $f', f'', f''', \text{ etc.}$ of a function $f(x)$ are the same type of function as $f$ itself.

**Example 5**

(a) If $f(x)$ is a polynomial, what type of function are the derivatives of $f$?

(b) If $f(x)$ is of the form $a \cos x + b \sin x$, where $a$ and $b$ are constants, what type of function are the derivatives of $f$?
(c) If \( f(x) \) is of the form \( e^{ax} \) where \( a \) is a constant, what type of function are the derivatives of \( f \)?

• Now consider the differential equation
  \[ y' + y = e^{2x}. \]
  • In light of example 5c, you might guess that \( y = ce^{2x} \), where \( c \) is a constant.
  • We can find the constant \( c \) by substituting \( ce^{2x} \) for \( y \) in the differential equation, and solving for \( c \):

**Important:** The inspection method gives just one solution of a differential equation, even though there may be infinitely many solutions. You’ll learn how to find all solutions later on.

**Example 6:** Determine by inspection a solution of the differential equation
\[ y'' + y = x^2 + x \]

**Hint:** If \( y \) is a quadratic (degree two) polynomial, then so is the left hand side of the differential equation above. So begin by letting \( y = ax^2 + bx + c \), where \( a, b \) and \( c \) are arbitrary constants. Substitute this \( y \) into the differential equation above, and solve for \( a, b, \) and \( c \).
Example 7: If $y = 3e^{t^2}$ is known to be the solution of the initial value problem

$$y' + p(t)y = 0, \quad y(0) = y_0, \quad y_0 \text{ constant},$$

what must the function $p(t)$ and the constant $y_0$ be?

Example 8: For what values of the constants $\alpha, y_0$, and integer $n$ is the function

$$y(t) = (4 + t)^{-1/2}$$

a solution of the initial value problem?

$$y' + \alpha y^n = 0, \quad y(0) = y_0$$