Examples of probability distribution functions for continuous random variables.

**Example 1:** A bus arrives at a bus stop every 30 minutes. Suppose that you show up at the stop at a random time. Let $T$ be your waiting time. Then $T$ is uniformly distributed on the interval $[0, 30]$, so $T$ has a *uniform* distribution. $T$ has probability distribution function

$$f(t) = \begin{cases} 1/30, & \text{if } 0 \leq t \leq 30 \\ 0, & \text{otherwise} \end{cases}$$

What is $P(10 \leq T \leq 20)$?

In general, if a random variable $X$ is uniformly distributed on the interval $[a, b]$, then $X$ has probability distribution function

$$f(t) = \begin{cases} 1/(b - a), & \text{if } a \leq t \leq b \\ 0, & \text{otherwise} \end{cases}$$

**Example 2:** A random variable $X$ has probability distribution function

$$f(x) = \begin{cases} k(2 - x), & \text{if } 0 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$$

What is the value of $k$?
Example 3: The exponential distribution. A radioactive source is emitting particles. Let $Y$ be the time (in minutes) until the next particle is emitted. Then for this particular radioactive isotope,

$$f(y) = \begin{cases} 
14e^{-14y}, & \text{if } y \geq 0 \\
0, & \text{if } y < 0.
\end{cases}$$

(a) What is $P(Y > t)$ if $t > 0$?

(b) What is $P(Y > s + t \mid Y > s)$?
Example 4: The most important continuous random variable. A random variable \( X \) with probability distribution function

\[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x-\mu)^2/2\sigma^2}
\]

is said to have a normal distribution with mean \( \mu \) and standard deviation \( \sigma \). In shorthand, this is written

\[ X \sim N(\mu, \sigma). \]

For example, IQ scores are normally distributed with mean 100 and standard deviation 10 and men’s heights are normally distributed with mean 69 inches and standard deviation 2.5 inches.

Expected value and variance of continuous random variables.

The expected value, or mean, of a continuous random variable \( X \) with probability distribution function \( f \) is

\[
\mu = E(X) = \int_{-\infty}^{\infty} x f(x) \, dx.
\]

The variance of \( X \) is

\[
\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx.
\]

Notice that

\[
\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x) \, dx - \int_{-\infty}^{\infty} 2x \mu f(x) \, dx + \int_{-\infty}^{\infty} \mu^2 f(x) \, dx
\]

\[
= \int_{-\infty}^{\infty} x^2 f(x) \, dx - 2\mu \int_{-\infty}^{\infty} x f(x) \, dx + \mu^2 \int_{-\infty}^{\infty} f(x) \, dx
\]

\[
= E(X^2) - 2\mu^2 + \mu^2
\]

\[
= E(X^2) - (E(X))^2.
\]

The standard deviation \( \sigma \) of \( X \) is the square root of the variance.

Example 5: Let \( X \) be uniformly distributed on \([a, b] \). Find \( E(X) \) and \( \sigma^2 \).
Example 6: Let $X$ have an exponential distribution, that is $X$ has probability distribution function

$$f(x) = \lambda e^{-\lambda x}, \lambda > 0.$$ 

Find $E(X)$ and $\sigma^2$. 