Math 112  Section 10  Lecture notes, 12/19/03

Pumping water:

1. A full cylindrical water tank has radius 1 feet and height 2 feet. It is buried in the ground so that its top is at ground level.
   (a) What is the volume of a circular cross-section of water with thickness $dy$?

   (b) What is the weight of the cross-section, if the density of water is 62.5 pounds per cubic foot?

   (c) If the displacement of the cross-section from ground level is $y$ feet, how much work is required to lift the cross section to ground level?

   (d) Now “add up” the work for each cross-section by integrating your part (c) answer. This gives the total work required to pump all the water to ground level.

2. A water tank is constructed by rotating the curve $y = 4 - x^2$, $0 \leq x \leq 2$, about the $y$–axis.
   (a) What is the weight of a circular cross-section of thickness $dy$ that is a distance of $y$ feet above the ground, as a function of $y$?
(b) How much work is required to lift the water in this cross section to height $y$ feet?

(c) How much work is required to fill the tank, if water is pumped in through a hole at the bottom of the tank?

**Propulsion:**

The weight of an object is the force exerted on it by the earth’s gravity. Thus, if a person weighs 100 lb on the surface of the earth, the earth’s gravity is pulling on that person with a force of 100 lb. It is a fundamental law of physics that the force the earth exerts on an object varies inversely as the square of its distance from the earth’s center. Thus an object’s weight $F(y)$ is related to its distance $y$ from the earth’s center by a formula of the form

$$F(y) = \frac{C}{y^2},$$

where $C$ is a constant of proportionality depending on the mass of the object, the mass of the earth, and the units of force and distance.

**3a.** Assuming that the earth is a sphere of radius 4000 miles, find the constant $C$ in the formula above for a satellite that weighs 6000 lbs on the earth’s surface.

**3b.** How much work is required to lift the satellite from a distance of $y$ miles from the earth’s center to a height of $y + dy$ miles above the earth’s center, if $dy$ is a tiny number?

**3c.** “Add up” the work amounts in part b from $y = 4000$ to $y = 5000$ by integrating your part b answer. This integral gives the total work required to lift the satellite from ground level to an orbital position 1000 miles above the earth’s surface.
Hooke’s law:

Hooke’s law states that under appropriate conditions a spring stretched \( x \) units beyond its natural position pulls back with a force

\[ F = kx \]

where \( k \) is a constant.

4. A spring whose natural length is 24 in. exerts a force of 5 lbs. when stretched 10 in. beyond its natural length.

(a) Find the spring constant \( k \).

(b) How much work is required to stretch the spring from a length of \( x \) in. beyond its natural length to a length of \( x + dx \) inches beyond its natural length, if \( dx \) is a tiny increment in length?

(c) How much work is required to stretch the spring from its natural length to a length of 42 in? Add up the work increments in part b from \( x = 0 \) to \( x = 18 \) by integrating your part b answer.

(d) How much work is required to stretch the spring from a length of 20 in. to a length of 30 in.?

Lifting a chain:

5a. A 100-ft length of chain weighing 15 lb/ft is dangling from a pulley. How much work does it take to lift a tiny portion of chain with length \( dy \) that is at a displacement of \( y \) feet onto the pulley?
5b. Add up the work required to lift each such portion of the chain onto the pulley, by integrating your part a answer. This gives the work required to wind the entire chain onto the pulley.